THEORY OF EQUATIONS

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Second Revised Edition

1954

S. CHAND & CO.,
Publishers
FOUNTAIN — DELHI

Published by
Gauri Shankar, Manager for
S. Chand & Co.,
Fountain, Delhi.

Price Rs. 5/-/-

Printed at
Baluja Press,
Fatehpuri, Delhi.

Preface to the First Edition

This book aims at providing elementary knowledge of the Theory of Equations and is planned to cover the syllabuses of the B. A. and B. Sc. (Pass and Honours) and M. A. examinations of Indian Universities. In our treatment of the subject we have laid particular stress on the practical aspect. We hope, this will help the student in mastering the elements of this useful branch of Mathematics. Miscellaneous questions taken from Mathematical Tripos (M. T.) and some other university examination papers have been appended at the end of the book.

In preparing this book, we have made free use of standard works on the subject and acknowledge with thanks the debt we owe to their authors.

For any suggestions for improvement from teachers and students, we shall feel very grateful.

R. B.
H. G.

Preface to the Second Edition

In this edition, we have made some minor changes in the text of the book and corrected some answers. We have also added questions set in recent years in some Indian and Foreign Universities. It has been no small encouragement to the authors to note that the book has found some sale abroad. It is hoped that this work will continue to serve the class of students for whom it is meant.

R. B.
H. G.

CONTENTS

THEORY OF EQUATIONS

CHAPTER I

GENERAL PROPERTIES OF EQUATIONS

§1. A book on the Theory of Equations aims at giving methods for the solution of algebraic equations involving one unknown quantity. The student is already familiar with equations of the first degree and also with quadratic equations.

The degree of an equation is the index of the highest power of the unknown quantity involved in the equation when it is expressed as a rational and integral function of the unknown quantity equated to zero.

For example, consider the equations

(*i*) $2x^3+5x^2+6x+8+\dfrac{4}{x}+\dfrac{3}{x^2}=0$;

(*ii*) $x^{\frac{2}{3}}+1=2x$.

The first equation on being freed from fractions can be written as

$$2x^5+5x^4+6x^3+8x^2+4x+3=0.$$

The degree of this equation is 5 ; x^5 being the highest power of the unknown quantity involved.

The second equation on being freed from radicals can be put in the form :

$$8x^3-13x^2+6x-1=0$$

Therefore, the degree of the equation is 3.

We could also put $x^{\frac{1}{3}}=y$ and get the degree.

Equations of the third degree are called cubic equations, and equations of the fourth degree are called biquadratic equations.

The equation

$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0,\ p_0\neq 0$;

is the general type of an equation of the nth degree. This may be written shortly as

$$f_n(x)=0$$

where n denotes the degree of the equation. The expression on the left hand side of this equation is called a Polynomial of degree n or a Rational Integral Function of degree n or a quintic.

EXERCISES—I.

Give the degree of the following equations :—

1. $3x^{\frac{1}{3}}+2x^{-\frac{1}{2}}=2x+1.$ **Ans.** 3rd.
2. $x^5+x^4-5x^3=6x^2-3x-2+4x^{-1}.$ **Ans.** 6th.
3. $x^{-\frac{3}{2}}+4x^{-2}=3x^{\frac{1}{2}}.$ **Ans.** 5th.
4. $\dfrac{x^2+x+2}{x+5}-\dfrac{x+4}{x^2-3}=\dfrac{3}{x}+\dfrac{x}{3}.$ **Ans.** 5th.

§2. Any value of x which makes $f_n(x)$ vanish is called a root of the equation $f_n(x)=0$.

Thus, 2 is a root of the equation :

$$x^3-6x^2+11x-6=0,$$

because for this value of x,

$$x^3-6x^2+11x-6=2^3-6.2^2+11.2-6=0.$$

Again, $2+\sqrt{3}$ is a root of the equation :

$$x^4-4x^3+6x^2-20x+5=0,$$

because for this value of x, the expression on the left side $=(2+\sqrt{3})^4-4(2+\sqrt{3})^3+6(2+\sqrt{3})^2-20(2+\sqrt{3})+5$ or 0.

An equation is said to be solved when all its roots are known.

EXERCISES—II.

1. Show that $4+\sqrt{3}$ is a root of the equation :

   $$x^5-8x^4+13x^3-27x^2+216x-351=0.$$

   Further show that $4-\sqrt{3}$ is also a root of the same equation.

2. Is 3 a root of the equation in Ex. I ? **Ans.** Yes.
3. Show that $2+\sqrt{3}i$ and $2-\sqrt{3}i$ are roots of the equation : $x^3-7x^2+19x-21=0.$

§3. We proceed to prove certain theorems concerning the roots of an equation.

Theorem 1. *If $f_n(x)$ be divisible by $(x-a)$, then a shall be a root of the equation $f_n(x)=0$.*

Conversely, if a be a root of the equation $f_n(x)=0$, then shall $f_n(x)$ be divisible by $(x-a)$.

Let $Q(x)$ denote the quotient, when $f_n(x)$ is divided by $(x-a)$ and R the remainder ; so that

$$f_n(x)\equiv(x-a)Q(x)+R.$$

In the first case, $f_n(x)$ is exactly divisible by $(x-a)$, hence

$$R=0.$$

Therefore, $f_n(x)$ vanishes when $x=a$, so that a is a root of the equation $f_n(x)=0$.

In the second case, since a is a root of the equation

$$f_n(x)=0,$$

we have $$f_n(a)\equiv(a-a)Q(a)+R=0\ ;$$

whence $$R=0.$$

Hence $f_n(x)$ is exactly divisible by $(x-a)$.

Theorem 2. *In an equation with* **real** *co-efficients, imaginary roots occur in pairs.*

Suppose $f_n(x)=0$ is an equation with real co-efficients and that it has an imaginary root $a+ib$, where a and b are real quantities, and b is not equal to zero.

It is required to prove that $a-ib$ is also a root of the equation.

Let $f_n(x)$ be divided by $(x-a)^2+b^2$. Denote the quotient by Q and the remainder, if any, by $Rx+R'$. Then

$$f_n(x)\equiv[(x-a)^2+b^2]\,Q+Rx+R'.$$

Put $x=a+ib$, then $f_n(x)$ vanishes, because $a+ib$ is root of the equation $f_n(x)=0$.

Also $(x-a)^2+b^2$ vanishes identically for this value of x.

Hence $R(a+ib)+R'=0.$

Equating the real and imaginary parts on the two sides of this relation, we have

$$Ra+R'=0 \text{ and } Rb=0.$$

Hence, b being not equal to zero,

$$R=0 \text{ and } R'=0.$$

Thus $f_n(x)$ is exactly divisible by $[(x-a)^2+b^2]$ *i.e.*, by $(x-a-ib)(x-a+ib)$.

Hence $a-ib$ is a root of the equation $f_n(x)=0$.

Ex. Where does the argument fail if the co-efficients in $f_n(x)$ are not all real ?

Theorem 3. *In an equation with* **rational** *co-efficients surd roots of the form $a \pm \sqrt{b}$ occur in pairs.*

Suppose $f_n(x)=0$ is an equation with rational co-efficients and that it has a root $a+\sqrt{b}$ where a and b are rational, b is not a perfect square of a rational quantity and is not equal to zero.

It is required to prove that $a-\sqrt{b}$ is also a root of the equation $f_n(x)=0$.

Let $f_n(x)$ be divided by $(x-a)^2-b$. Denote the quotient by Q and the remainder, if any, by $Rx+R'$. Then

$$f_n(x) \equiv [(x-a)^2-b]Q+Rx+R'.$$

In this identity, put $x=a+\sqrt{b}$, then since $f_n(a+\sqrt{b})=0$ and $(x-a)^2-b$ vanishes identically when $x=a+\sqrt{b}$ we have

$$R(a+\sqrt{b})+R' \equiv 0.$$

Equating the rational and irrational parts on the two sides of this relation, we have

$$Ra+R'=0 \text{ and } R\sqrt{b}=0$$

whence $$R=0 \text{ and } R'=0.$$

This proves that $f_n(x)$ is exactly divisible by $[(x-a)^2-b]$.

Hence $a-\sqrt{b}$ is a root of $f_n(x)=0$.

Ex. Where does the argument fail if the co-efficients in $f_n(x)$ are not all rational ?

Example. Solve the equation

$$x^5-x^4+8x^2-9x-15=0,$$

one root being $\sqrt{3}$ and another $1-2i$.

The co-efficients being all real and rational, $1+2i$ and $-\sqrt{3}$ must also be roots of the given equation.

Hence $x^5-x^4+8x^2-9x-15$ must be exactly divisible by each of the factors :

$$(x-\sqrt{3}),\ (x+\sqrt{3}),\ (x-1+2i) \text{ and } (x-1-2i);$$

i.e., by the quadratic factors (x^2-3) and (x^2-2x+5).

We have in fact

$$x^5-x^4+8x^2-9x-15 \equiv (x^2-3)(x^2-2x+5)(x+1)$$

as may be seen on actual division.

Now since $(x+1)$ *i.e.*, $[x-(-1)]$ is a factor of the expression on the left side, -1 must be a root of the given equation.

EXERCISES—III.

Solve :—

1. $6x^4-13x^3-35x^2-x+3=0$, one root being $2-\sqrt{3}$.
   Ans. $2\pm\sqrt{3},\ -1/3,\ -3/2$.
2. $x^4+4x^3+6x^2+4x+5=0$, one root being i.
   Ans. $\pm i,\ -2\pm i$.
3. $3x^3-4x^2+x+88=0$, one root being $2+\sqrt{-7}$.
   Ans. $2\pm\sqrt{-7},\ -8/3$.
4. $x^4+2x^3-5x^2+6x+2=0$, one root being $-2+\sqrt{3}$.
   Ans. $-2\pm\sqrt{3},\ 1\pm i$.
5. $2x^5+x^4-6x^3-3x^2+4x+2=0$, one root being $-1/2$.
   Ans. $-1/2,\ \pm 1,\ \pm\sqrt{2}$.
6. $x^4-16x^3+86x^2-176x+105=0$, two roots being 1 and 7.
   Ans. 1, 3, 5, 7.

§ 4. *To form an equation with given roots.*

We first observe that if any quantity h be a root of the equation $f(x)=0$, then $f(x)$ is divisible by $(x-h)$ without a remainder.

Now let $\alpha, \beta, \gamma,$.........be the given roots.

Then the required equation is

$$c(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots=0,$$

c being a constant.

The factors on the left hand side of the above equation can be multiplied together and the terms in the product arranged in descending powers of x.

Example 1. Form an equation having 1, 2, 3, -1, -2 and -3 for its roots.

The required equation is

$$(x-1)(x-2)(x-3)\,[x-(-1)]\,[x-(-2)][x-(-3)]=0,$$

i.e., $$(x^2-1)(x^2-4)(x^2-9)=0,$$

or $$x^6-14x^4+49x^2-36=0.$$

Example 2. Form an equation of the lowest degree with rational co-efficients having $\sqrt{5+2\sqrt{6}}$ for one of its roots.

Let $x=\sqrt{5+2\sqrt{6}}$.

Squaring, we have

$$x^2=5+2\sqrt{6}, \text{ or } x^2-5=2\sqrt{6}.$$

Therefore

$$(x^2-5)^2=24. \text{ or } x^4-10x^2+1=0$$

is the required equation.

Alternative method.

Since in an equation with rational co-efficients, surd roots occur in pairs, the two pairs of roots are :

$$\pm\sqrt{5+2\sqrt{6}} \text{ and } \pm\sqrt{5-2\sqrt{6}}.$$

Corresponding to these roots, we have the factors :

$$(x-\sqrt{5+2\sqrt{6}}),\ (x+\sqrt{5+2\sqrt{6}}),$$

and $(x-\sqrt{5-2\sqrt{6}})$, $(x+\sqrt{5-2\sqrt{6}})$.

Hence the required equation is

$$(x^2-5-2\sqrt{6})\,(x^2-5+2\sqrt{6})=0,$$

i.e.
$$x^4-10x^2+1=0.$$

EXERCISES—IV

1. Form the equation whose roots are :

   (*i*) $\frac{2}{3}, \frac{3}{2}, \pm\sqrt{3}$;
   Ans. $6x^4-13x^3-12x^2+39x-18=0.$

   (*ii*) 0, 1, 2, 3, 4, 5 ;
   Ans. $x^6-15x^5+85x^4-225x^3+274x^2-120x=0.$

   (*iii*) $a\pm b, -a\pm b$.
   Ans. $x^4-2(a^2+b^2)x^2+(a^2-b^2)^2=0.$

2. Form an equation of the lowest degree with real and rational co-efficients, one of its roots being

   (*i*) $\sqrt{3}+\sqrt{-2}$; (*ii*) $\sqrt{2}+\sqrt{3}+i$.
   Ans. (*i*) $x^4-2x^2+25=0.$
   (*ii*) $x^8-16x^6+88x^4+192x^2+144=0.$

3. Form a rational cubic which shall have 1, $3+2i$ for its roots. **Ans.** $x^3-7x^2+19x-13=0.$

4. Form a rational equation which shall have $\sqrt{p}+\sqrt{q}$ for a root. **Ans.** $x^4-2(p+q)x^2+(p-q)^2=0.$

5. Form a rational equation having $1+5i$ and $5-i$ for two of its roots. **Ans.** $x^4-12x^3+72x^2-312x+676=0.$

§5. *Every equation of the nth degree has n roots and no more.*

Let $f_n(x) \equiv p_o x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots\ldots+p_{n-1}x+p_n=0$ represent the given equation.

We assume the proposition* that 'Every equation has a root'.

Hence the equation $f_n(x)=0$ must have a root real or imaginary.

Let it be denoted by a_1.

Then $f_n(x)$ is divisible by $(x-a_1)$;

---

*The proof of this proposition is too advanced to be included here.

so that $f_n(x) \equiv (x-a_1)\, f_{n-1}(x)$,
where $f_{n-1}\,(x)$ is a function of $(n-1)$ dimensions.

Again, the equation $f_{n-1}(x)=0$ must have a root real or imaginary.

Let it be denoted by a_2.

Then $f_{n-1}(x)$ is divisible by $(x-a_2)$, so that

$f_{n-1}(x) \equiv (x-a_2)\, f_{n-2}(x)$,
where $f_{n-2}(x)$ is a function of $(n-2)$ dimensions.

Thus, we have

$$f_n(x) \equiv (x-a_1)\,(x-a_2)\, f_{n-2}(x).$$

Reasoning in this manner, we see that

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)$$

is a factor of $f_n(x)$.

Let $f_n(x) \equiv (x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)\ \mathrm{Q} \quad \ldots \quad (i)$

As both $f_n(x)$ and $(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)$ are of the nth degree, Q must be a constant and equal to p_o.

Now, the right hand side of this identity vanishes when $x=a_1, a_2, a_3,\ldots\ldots\ldots,a_n$. The equation $f_n(x)=0$ has therefore n roots.

Also the equation cannot have more than n roots, for if x has any value c different from any of the quantities a_1, a_2, $a_3,\ldots\ldots\ldots a_n$, then all the factors on the right hand side of (i) are different from zero, so that $f_n(x)$ does not vanish when $x=c$. Therefore c is not a root of the equation $f_n(x)=0$. It is not, however, neecessary that the quantities $a_1, a_2, a_3,\ldots\ldots\ldots,a_n$ be all different.

§6. We shall now present an exposition of certain contracted processes and prove some general theorems concerning polynomials. These shall be of use to us in the succeeding chapters.

§7. *To find the quotient and the remainder when a polynomial is divided by a binomial.*

Let $\mathrm{Q} \equiv a_o x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} + \ldots\ldots + a_{n-1}$

denote the quotient and R the remainder, when the polynomial

$$f_n(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n$$

is divided by the binomial $x-h$. Then

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} \ldots\ldots + p_n$$
$$\equiv (x-h)\,(a_0x^{n-1} + a_1x^{n-2} + a_2x^{n-3} + \ldots\ldots + a_{n-1}) + \mathrm{R}.$$

Equating co-efficients of like powers of x on the two sides of this identity, we have

$$\begin{array}{lll} & p_0 = a_0, & i.e. \;\; a_0 = p_0, \\ & p_1 = a_1 - a_0h, & i.e. \;\; a_1 = p_1 + a_0h, \\ & p_2 = a_2 - a_1h, & i.e. \;\; a_2 = p_2 + a_1h. \\ & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ & p_{n-1} = a_{n-1} - a_{n-2}h & i.e. \, a_{n-1} = p_{n-1} + a_{n-2}h, \\ \text{and} & p_n = \mathrm{R} - a_{n-1}h & i.e. \;\; \mathrm{R} = p_n + a_{n-1}h. \end{array}$$

These equations provide a ready means of calculating in succession the coefficients $a_0, a_1, a_2 \ldots\ldots a_{n-1}$ in the quotient Q, and the remainder R. We write the series of operations in the following manner :

$$h \left| \begin{array}{ll} p_0,\; p_1,\; p_2,\; p_3, \ldots\ldots, p_{n-1}, & p_n \\ \quad a_0h,\; a_1h,\; a_2h, \ldots\ldots, a_{n-2}h, & a_{n-1}h \\ \hline a_0,\; a_1,\; a_2,\; a_3, \ldots\ldots, a_{n-1} & \mathrm{R} \end{array} \right.$$

In the first row are written the co-efficients in the given polynomial $f_n(x)$. The first term in the third row is the same as the first term in the first row. The first term in the second row is obtained by multiplying $a_0 (= p_0)$ by h. The product is placed under p_1 and added to it in order to get a_1. This, in its turn, is multiplied by h. The product is placed under p_2 and added to it in order to get a_2. This process of multiplication and addition is carried on. The third row, then, supplies us with the coefficients in the quotient and also the remainder R.

Example. Find the quotient and the remainder when

$$2x^5 - 3x^3 + 2x^2 - 5x - 46 \text{ is divided by } (x-2).$$

The work of calculation is arranged as follows :—

$$2 \left| \begin{array}{l} 2+0-3+\;2-\;5-46 \\ \quad\; 4+8+10+24+38 \\ \hline 2+4+5+12+19 \,|\, -8 \end{array} \right.$$

Thus the quotient is $2x^4+4x^3+5x^2+12x+19$ and the remainder is -8.

Note it **Caution.** When any term is absent the place of its co-efficient is taken by a zero.

EXERCISES—V

Find the quotient and the remainder, when

1. $x^7-6x^4+3x^3-2x^2+3$ is divided by $x-1$.
   Ans. $x^6+x^5+x^4-5x^3-2x^2-4x-4$; -1.
2. $x^5-4x^4+3x^2-6x+2$ is divided by $x+5$.
   Ans. $x^4-9x^3+45x^2-222x+1104$; -5518.
3. $x^4+10x^3+39x^2+76x+65$ is divided by $x+4$.
   Ans. $x^3+6x^2+15x+16$; 1.
4. Divide $2x^4-13x^2+10x-.9$ by $(x-2)$, the quotient by $(x-3)$ and the resulting quotient by $(x-4)$. Give the successive quotients and remainders. Hence write the given expression in the form :
   $(x-2)[(x-3)\{(x-4)\ Q_3+R_3\}+R_2]+R_1$.

Solution. The process is arranged as follows :—

$$
\begin{array}{c|l}
 & 2+0-13+10-19 \\
2 & \underline{\quad +4+8-10+0} \\
 & 2+4-5+0 \mid -19=R_1 \\
3 & \underline{\quad +6+30+75} \\
 & 2+10+25 \mid +75=R_2 \\
4 & \underline{\quad +8+72} \\
 & 2+18 \mid +97=R_3
\end{array}
$$

The successive quotients are

$2x^3+4x^2-5x$; $2x^2+10x+25$; $2x+18$.

The successive remainders are : -19 ; 75 ; 97.

Hence $2x^4-13x^2+10x-19$

$\equiv (x-2)\ [(x-3)\{(x-4)(2x+18)+97\}+75]-19$.

5. Divide $5x^4-3x^3+x^2+x+1$ four times in succession by $(x-1)$. Hence write the given expression in the form $a_1(x-1)^4+a_2(x-1)^3+a_3(x-1)^2+a_4(x-1)+a_5$.
   Ans. $a_1=5$, $a_2=17$, $a_3=22$, $a_4=14$, $a_5=5$.

§8. In the preceding article, we have found the quotient and the remainder when a polynomial is divided by a binomial. This helps us in finding the value of a polynomial for a given value of the variable.

Suppose we have to find the value of $f(x)$ when $x=h$.

Divide $f(x)$ by $(x-h)$. Let Q denote the quotient and R the remainder ; then $f(x)\equiv(x-h)\mathrm{Q}+\mathrm{R}$.

Put $x=h$ in this identity; then $f(h)=\mathrm{R}$.

Thus it is seen that $f(h)$ is the remainder obtained when $f(x)$ is divided by $(x-h)$.

Example. Find the value of $4x^4-3x^3-2x^2-3$ when $x=-3$.

Here, we divide the given polynomial by $(x+3)$ as follows :—

$$\begin{array}{r|l} -3 & 4-\ 3-\ 2+\ \ 0-\ \ 3 \\ & \quad -12+45-129+387 \\ \hline & 4-15+43-129\,|+384 \end{array}$$

Thus 384 is the required value.

Ex. Find the value of $x^5-5x^4+10x^3-10x^2+5x-1$ when $x=3, 2, 1, 0, -1, -2, -3$.

Ans. 32, 1, 0, -1, -32, -243, -1024.

§9. **Synthetic Division.** *To find the quotient and the remainder when one polynomial is divided by another.*

Suppose we have to divide the polynomial

$$\mathrm{A}\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_n,$$

by the polynomial

$$\mathrm{B}\equiv b_0x^m+b_1x^{m-1}+b_2{}^{m-2}+\ldots\ldots+b_m,.$$

and get the quotient in descending powers of x to $(k+1)$ terms and the corresponding remainder.

Let the quotient be

$$\mathrm{Q}\equiv q_0x^{n-m}+q_1x^{n-m-1}+q_2x^{n-m-2}+\ldots\ldots+q_kx^{n-m-k},$$

and the corresponding remainder be

$$\mathrm{R}\equiv r_1x^{n-k-1}+r_2x^{n-k-2}+r_3x^{n-k-3}+\ldots\ldots$$

where the remainder has either $(n-k)$ or m terms, whichever is more.

Then we have $A \equiv BQ+R$.

Comparing the co-efficients of the powers of x on the two sides of this identity and taking a_t to be zero when $t>n$, we get

$$\begin{aligned}
&a_0=b_0q_0\text{ ; therefore } q_0=a_0/b_0\text{ ;}\\
&a_1=b_1q_0+b_0q_1\text{ ; therefore } q_1=(a_1-b_1q_0)/b_0\text{ ;}\\
&a_2=b_2q_0+b_1q_1+b_0q_2\text{ ; therefore } q_2=(a_2-b_2q_0-b_1q_1)/b_0\text{ ;}\\
&\qquad\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_k=b_kq_0+b_{k-1}q_1+b_{k-2}q_2+\ldots\ldots+b_0q_k\text{ ;}\\
&\qquad\text{therefore } q_k=(a_k-b_kq_0-b_{k-1}q_1-\ldots\ldots-b_1q_{k-1})/b_0\text{ ;}\\
&a_{k+1}=b_{k+1}q_0+b_kq_1+\ldots\ldots+b_1q_k+r_1\text{ ;}\\
&\qquad\text{therefore } r_1=a_{k+1}-b_{k+1}q_0-b_kq_1-\ldots\ldots-b_1q_k\text{ ;}\\
&a_{k+2}=b_{k+2}q_0+b_{k+1}q_1+\ldots\ldots+b_2q_k+r_2\text{ ;}\\
&\qquad\text{therefore } r_2=a_{k+2}-b_{k+2}q_0-b_{k+1}q_1-\ldots\ldots-b_2q_k\text{ ;}
\end{aligned}$$

and so on, where $b_t=0$, when $t>m$.

These equations provide us with a ready means of calculating the q's and r's.

The operations are presented in the following convenient form :—

$$\begin{array}{c|llllll}
 & a_0+ & a_1+ & a_2+\ldots\ldots+ & a_k+a_{k+} & & +\ldots\ldots\\
-b_1 & & -b_1q_0 & -b_2q_0-\ldots\ldots & -b_3q_0 & -b_kq_0 & -\ldots\ldots\\
-b_2 & & & -b_1q_1-\ldots\ldots & -b_{k-1}q_1 & -b_kq_1 & -\ldots\ldots\\
-b_3 & & & & -b_{k-2}q_2 & -b_{k-1}q_2 & -\ldots\ldots\\
\cdots & & & & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\\
-b_m & & & & -b_1q_{k-1} & -b_2q_{k-1} & -\ldots\ldots\\
 & & & & & -b_1q_k & -\ldots\ldots\\
0 & & & & & & \cdots\cdots\cdots \quad X\\
\hline
b_0 & a_0+b_0q_1 & +b_0q_2 & +\ldots\ldots & +b_0q_k & +r_1+\ldots\ldots & \\
 & \text{(leading row)} & & & & \text{remainder row} & \\
\hline
 & q_0+q_1 & +q_2 & +\ldots\ldots & +q_k & & \\
 & \text{(quotient row)} & & & & &
\end{array}$$

The successive co-efficients $a_0, a_1, a_2, \ldots\ldots, a_n$ in the 'dividend' are written to the right of the vertical line OY. To the left, in column, are written the successive co-efficients b_1, b_2, $b_3 \ldots, b_m$ in divisor with their signs changed, the co-efficient b_0

being written further down against the 'leading row', with its proper sign. The first element from the left in the 'leading row' is the same as in the dividend *viz.*, $a_0(=b_0q_0)$. On division by b_0, this gives q_0 which is written under a_0, in the 'quotient row'. q_0 is multiplied in succession by $-b_1, -b_2, -b_3 \ldots\ldots -b_m$ and the products are written under $a_1, a_2, a_3, \ldots\ldots, a_m$ in a row. The second element b_0q_1 in the 'leading row' is obtained by adding the two elements a_1 and $-b_1q_0$ above it. This on division by b_0 gives q_1 which is the second element in the 'quotient row'. q_1 is in its turn, multiplied by $-b_1, -b_2, -b_3, \ldots\ldots, -b_m$ in succession and the products are written under the elements of the previous row leaving out one element from the left. Adding up a_2, $-b_2q_0$ and $-b_1q_1$, we get b_0q_2 in the 'leading row', giving q_2 in the 'quotient row'. This process of addition, division by b_0 and multiplication by $-b_1, -b_2, -b_3 \ldots -b_m$ is carried on until the required number of terms is obtained in the quotient. The last element b_0q_k in the 'leading row' is obtained by adding the elements

$$a_k,\ -b_kq_0,\ -b_{k-1}q_1,\ -b_{k-2}q_2, \ldots\ldots, \text{ and } -b_1q_{k-1}.$$

On division by b_0, this gives the last required element q_k in the 'quotient row'. q_k is multiplied by $-b_1, -b_2, -b_3, \ldots\ldots, -b_m$ and the products are written under the elements of the preceding row leaving out one element from the left as heretofore. Adding up the remaining columns of numbers now, we get the elements in the 'remainder row' *viz.*, $r_1, r_2, r_3 \ldots\ldots$ We give a few examples to illustrate the points involved in the above process.

Example 1. Get four terms in the quotient and the corresponding remainder in the division of $4x^3+2x^2-2x-1$ by $2x^2+3x-3$.

The process is exhibited thus :

$$\begin{array}{r|rrrr|rr}
 & 4 & +2 & -2 & -1 & & \\
-3 & & -6 & +6 & & & \\
3 & & & +6 & -6 & & \\
 & & & & -15 & +15 & \\
 & & & & & +33 & -33 \\
\hline
2 & 4 & -4 & +10 & -22 & 48 & -33 \\
\hline
 & 2 & -2 & +5 & -11 & & \\
\end{array}$$

Hence the quotient is $2x-2+5x^{-1}-11x^{-2}$ and the remainder is $48x^{-1}-33x^{-2}$.

Example 2. Find the co-efficients of x^{-4} and x^{-5} in the quotient of $f'(x)$ by $f(x)$, when $f(x) \equiv x^4+3x^2-3x+4$.

In this example, two points require special notice.

(*i*) The co-efficient of the first term in the divisor is unity. The elements of the 'quotient row' will, therefore, be identical with those of the 'leading row'. The two rows may thus be merged into one.

(*ii*) We require the co-efficients of x^{-4} and x^{-5} in the quotient. Therefore, the quotient need be calculated upto 5 terms only. The columns beyond the fifth need not be exhibited, because they supply only the co-efficients in the remainder.

The dividend is $f'(x) \equiv 4x^3+6x-3$.

Now, the process of division will be arranged thus :

$$
\begin{array}{r|rrrrr}
 & 4 & +0 & +6 & -3 & \\
0 & & +0 & -12 & +12 & -16 \\
-3 & & & +0 & +0 & +0 \\
3 & & & & +0 & +18 \\
-4 & & & & & +0 \\
\hline
1 & 4 & +0 & -6 & +9 & +2 \\
\hline
\end{array}
$$

Thus the required co-efficients are 9 and 2 respectively.

Example 3. Find the quotient and the remainder when $4x^7-2x^6-10x^5-25x^4-40x^2+26x-21$ is divided by
$$2x^3-5x^2+2x-7.$$

By the method of synthetic division

$$
\begin{array}{r|rrrrr|rrr}
 & 4 & -2 & -10 & -25 & +0 & -40 & +26 & -21 \\
5 & & +10 & -4 & +14 & & & & \\
-2 & & & +20 & -8 & +28 & & & \\
7 & & & & +15 & -6 & +21 & & \\
 & & & & & -10 & +4 & -14 & \\
 & & & & & & +30 & -12 & +42 \\
\hline
2 & 4 & +8 & +6 & -4 & +12 & +15 & +0 & +21 \\
\hline
 & 2 & +4 & +3 & -2 & +6 & & & \\
\hline
\end{array}
$$

the required quotient is $2x^4+4x^3+3x^2-2x+6$, and the remainder is $15x^2+21$.

EXERCISES VI

1. Find the quotient and the remainder, when

(*i*) $15x^7-16x^6+30x^5-3x^4-5x^3-2x^2+5x+7$ is divided by x^2-x+1 ;

(*ii*) $27x^4-21x^3+x^2-2x+2$ is divided by $3x-1$;

(*iii*) $3x^5-4x^3-2x+5$ is divided by x^3+x^2-4x-3.

Ans. (*i*) $15x^5-x^4+14x^3+12x^2-7x-21$; $-9x+28$.

(*ii*) $9x^3-4x^2-x-1$; 1.

(*iii*) $3x^2-3x+11$; $-14x^2+33x+38$.

2. Calculate to five terms the quotient and the corresponding remainder, when

(*i*) $5x^2-3x+1$ is divided by $x^5-3x^4+x^3-x-1$;

(*ii*) x^2+2x+1 is divided by x^3+3x^2+3x+1.

Ans. (*i*) $5x^{-3}+12x^{-4}+32x^{-5}+84x^{-6}+225x^{-7}$,

$608x^{-3}-181x^{-4}+116x^{-5}+309x^{-6}+225x^{-7}$.

(*ii*) $x^{-1}-x^{-2}+x^{-3}-x^{-4}+x^{-5}$, $-x^{-3}-2x^{-4}-x^{-5}$.

§ **10.** *To find the H. C. F. of two given polynomials.*

The student is already familiar with the process of finding the highest common factor of two polynomials. Much labour and space may be saved by following the abbreviated method employed below.

Example 1. Find the H.C.F. of
$8x^4+4x^3-18x^2+11x-2$ and $32x^3+12x^2-36x+11$.

The process may be arranged as follows :—

	$8+4-18+11-2$		
	4		
1	$32+16-72+44-8$	$32+12-36+11$	8
	$32+12-36+11$	$32-32+8$	
	$4-36+33-8$	$44-44+11$	11
	8	$44-44+11$	
1	$32-288+264-64$	$\times$	
	$32+12-36+11$		
	$-75)\ -300+300-75$		
	$4-4+1$		

The required H. C. F. is $4x^2-4x+1$.

In the above process the coefficients have been arranged in two parallel columns and the quotients are placed to the right and left. The process of division is carried on until the remainder is of a lower degree than the divisor.

The process is exhibited in its fullness thus :

$$
\begin{array}{r}
8x^4+4x^3-18x^2+11x-2 \\
4 \\
32x^3+12x^2-36x+11)\overline{32x^4+16x^3-\ 72x^2+\ 44x-\ 8}(x \\
\underline{32x^4+12x^3-\ 36x^2+\ 11x\qquad} \\
4x^3-\ 36x^2+\ 33x-8 \\
8 \\
\overline{32x^3-288x^2+264x-64}(1 \\
\underline{32x^3+\ 12x^2-\ 36x+11} \\
-75)\overline{-300x^2+300x-75} \\
4x^2-4x+1
\end{array}
$$

$$
\begin{array}{r}
4x^2-4x+1)\overline{32x^3+12x^2-36x+11}(8x+11 \\
\underline{32x^3-32x^2+\ 8x\qquad} \\
44x^2-44x+11 \\
\underline{44x^2-44x+11} \\
\times
\end{array}
$$

Example 2. Find the H. C. F. of

$$x^5-x^3+4x^2-3x+2 \text{ and } 5x^4-3x^2+8x-3.$$

The process is arranged as follows :

	$1+0-1+4-3+2$		
	5		
1	$5+0-5+20-15+10$	$5+0-3+8-3$	5
	$5+0-3+8-3$	$5-30+30-25$	
	$-2)-2+12-12+10$	$3)\ 30-33+33-3$	
1	$1-6+6-5$	$10-11+11-1$	10
	$1-1+1$	$10-60+60-50$	
-5	$-5+5-5$	$49)\ 49-49+49$	
	$-5+5-5$	$1-1+1$	
	$\times$		

Therefore the required H. C. F. $=x^2-x+1$.

Example 3. Find the H. C. F. of

$$x^5+x^4-4x+2 \text{ and } x^3-2x+1.$$

The process is arranged as follows :

$$
\begin{array}{r|r|r|l}
1 & 1+1+0+0-4+2 & 1+0-2+1 & 1 \\
 & 1+0-2+1 & 1-1+0 & \\
\hline
1 & 1+2-1-4+2 & 1-2+1 & 1 \\
 & 1+0-2+1 & 1-1+0 & \\
\hline
2 & 2+1-5+2 & -1)-1+1 & \\
 & 2+0-4+2 & 1-1 & \\
\hline
1 & 1-1+0 & & \\
 & 1-1 & & \\
\hline
 & \times & &
\end{array}
$$

Therefore the required H. C. F. is $x-1$.

EXERCISES—VII

Find the H. C. F. of

1. $3x^3-7x^2-18x-8$ and $2x^3-3x^2-17x-12$
Ans. x^2-3x-4.

2. $22x^6-78x^5-16x^2$ and $2x^5-78x^2-44x$.
Ans. $2x(x^2-3x-2)$.

3. $12x^4a^2+54x^3a^3+6x^2a^4-72xa^5$ and $2x^6a+15x^5a^2+40x^4a^3+45x^3a^4+18x^2a^5$.
Ans. $ax(2x+3a)$.

4. $4x^4+11x^3+27x^2+17x+5$ and $6x^4+14x^3+36x^2+14x+10$.
Ans. x^2+2x+5.

5. $4x^4-16x^3+108$ and $6x^5-14x^3-40x^2+36$.
Ans. $2(x^2+2x+3)$.

§11. *To find the value of $f_n(x+h)$ when $f_n(x)$ is a polynomial of degree n.*

We have to consider in this article, the change of form of the polynomial $f_n(x)$ corresponding to an increase or decrease in the variable.

Let $f_n(x)\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n$.

Also let $f_n(x+h)\equiv P_0x^n+P_1x^{n-1}+P_2\,x^{n-2}+\ldots\ldots+P_{n-1}x+P_n \quad (i)$

where the P's have to be determined.

Putting zero for x in (i) and its successive differential coefficients with respect to x, we have

$$P_n \;= f_n(h),$$

$$P_{n-1} = \frac{f'_n(h)}{2\,!},$$

$$P_{n-2} = \frac{f''_n(h)}{2\,!},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_{n-r} = \frac{f_n^{(r)}(h)}{r\,!},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_1 \;= \frac{f_n^{(n-1)}(h)}{(n-1)\,!},$$

and $$P_0 \;= \frac{f_n^{(n)}(h)}{n\,!};$$

where $f_n^{(r)}(h)$ indicates that $f_n(x)$ has been differentiated r times and x is then replaced by h. We thus have

$$f_n(x+h) = \frac{f_n^{(n)}(h)}{n\,!}x^n + \frac{f_n^{(n-1)}(h)}{(n-1)\,!}x^{n-1} + \cdots\cdots$$

$$\cdots\cdots + \frac{f_n^{(r)}(h)}{r\,!}x^r + \ldots + f_n(h),$$

or more shortly

$$f_n(x+h) = \sum_{r=0}^{n} \frac{f_n^{(r)}(h)}{r\,!}x^r.$$

Example. If $f(x)=x^4+10x^3+39x^2+76x+65$, find $f(x-4)$.

We have $f(x)=x^4+10x^3+39x^2+76x+65$,

$$f'(x)=4x^3+30x^2+78x+76,$$

$$f''(x)=12x^2+60x+78,$$

$$f'''(x)=24x+60, \text{ and } f^{iv}(x)=24.$$

Substituting -4 for x in these equations, we get

$$P_0=\frac{f^{iv}(-4)}{4\,!}=1,\;\; P_1=\frac{f'''(-4)}{3\,!}=-6,\;\; P_2=\frac{f''(-4)}{2\,!}=15,$$

$$P_3 = \frac{f'(-4)}{1\,!} = -12 \text{ and } P_4 = f(-4) = 1.$$

Hence $f(x-4) = x^4 - 6x^3 + 15x^2 - 12x + 1$.

EXERCISES—VIII.

1. If $f(x) = 2x^4 - 13x^2 + 10x - 19$, find the value of
(*i*) $f(x+1)$, (*ii*) $f(x-1)$, (*iii*) $f(x+3)$, (*iv*) $f(x-5)$.
Ans. (*i*) $2x^4 + 8x^3 - x^2 - 8x - 20$;
(*ii*) $2x^4 - 8x^3 - x^2 + 28x - 40$;
(*iii*) $2x^4 + 24x^3 + 95x^2 + 148x + 56$;
(*iv*) $2x^4 - 40x^3 + 287x^2 - 860x + 856$.

2. If $f(x) = x^5 - 5x^2 - 3x + 2$, find the value of
(*i*) $f(x-2)$, (*ii*) $f(x-3)$, (*iii*) $f(x+3)$.
Ans. (*i*) $x^5 - 10x^4 + 40x^3 - 85x^2 + 97x - 44$.
(*ii*) $x^5 - 15x^4 + 90x^3 - 275x^2 + 432x - 277$.
(*iii*) $x^5 + 15x^4 + 90x^3 + 265x^2 + 372x + 191$.

3. If $f(x) = ax^8 + bx^5 + cx + d$, find $f(x+h) - f(x-h)$.
Ans. $2[8ahx^7 + 56ah^3x^5 + 5bhx^4 + 56ah^5x^3 + 10bh^3x^2 + 8ah^7x + bh^5 + ch]$.

§**12.** The method of the previous article becomes tedious in practice. When the co-efficients are all numerical, the calculation is best made by Horner's process :

Let $f_n(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots + p_{n-1}x + p_n$.
Suppose $f_n(x+h) \equiv P_0x^n + P_1x^{n-1} + P_2x^{n-2} + \dots + P_{n-1}x + P_n$.

To determine the P's, put $(x-h)$ for x in this identity, then

$$f_n(x) \equiv P_0(x-h)^n + P_1(x-h)^{n-1} + P_2(x-h)^{n-2} + \dots\dots + P_{n-1}(x-h) + P_n.$$

Hence P_n is the remainder obtained by dividing $f_n(x)$ by $(x-h)$.

Also the quotient resulting from division is

$$P_0(x-h)^{n-1}+P_1(x-h)^{n-2}+P_2(x-h)^{n-3}+\ldots\ldots$$
$$+P_{n-2}(x-h)+P_{n-1}\,;$$

so that P_{n-1} is the remainder obtained by dividing this quotient by $(x-h)$. The other P's are similarly obtained by dividing the quotients successively by $(x-h)$. Also $P_0=p_0$.

Example. If $f(x)=x^4+10x^3+39x^2+76x+65$, find $f(x-4)$. Here, we divide $f(x)$ successively by $(x+4)$, thus

$$\begin{array}{r|l}
 & 1+10+39+76+65 \\
-4 & \quad -\;4-24-60-64 \\
\hline
 & 1+\;6+15+16\;\;+1=P_4 \\
 & \quad -\;4-\;8-28 \\
\hline
 & 1+\;2+\;7\;\;-12=P_3 \\
 & \quad -\;4+\;8 \\
\hline
 & 1-2\;\;+15=P_2 \\
 & \quad -\;4 \\
\hline
 & 1-\;6=P_1.
\end{array}$$

The successive remainders are : 1, -12, 15, -6 and 1.

Hence $f(x-4)=x^4-6x^3+15x^2-12x+1$.

EXERCISES IX

1. If $f(x)=2x^4-13x^2+10x-19$, find the value of (*i*) $f(x+1)$, (*ii*) $f(x+3)$, (*iii*) $f(x-5)$.

Ans. See 1, VIII.

3. Given $f(x)=x^4-12x^3+17x^2-0x+7$; obtain (*i*) $f(x+3)$, (*ii*) $f(x-1)$, (*iii*) $f(x-2)$.

Ans. (*i*) $x^4-37x^2-123x-110$,
(*ii*) $x^4-16x^3+59x^2-83x+46$.
(*iii*) $x^4-20x^3+113x^2-253x+205$.

A Note on Binomial co-efficients.

In many algebraic processes it will be found convenient to write the general polynomial of the nth degree in the form :

$${}^nC_0p_0x^n+{}^nC_1p_1x^{n-1}+{}^nC_2p_2x^{n-2}+\ldots\ldots+{}^nC_rp_rx^{n-r}+\ldots\ldots\ldots$$
$$+{}^nC_np_n$$

in which each term is in addition to its literal co-efficient multiplied with the numerical co-efficient of the corresponding term in the expansion of $(x+1)^n$ by the Binomial Theorem. The $(r+1)$th term in the polynomial is ${}^nC_r p_r x^{n-r}$ and the polynomial can be written shortly as

$$\phi_n(x) \equiv \sum_{r=0}^{n} {}^nC_r p_r x^{n-r}$$

Thus, we have

$$\begin{aligned}
&\phi_0(x)=p_0,\\
&\phi_1(x)=p_0x+p_1,\\
&\phi_2(x)=p_0x^2+2p_1x+p_2,\\
&\phi_3(x)=p_0x^3+3p_1x^2+3p_2x+p_3,\\
&\phi_4(x)=p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4,
\end{aligned}$$

and so on.

One advantage of writing the polynomial in this form is that $\phi_n(x)$ is very easily differentiated and $\phi_n(x+h)$ is readily found by the method of §15, Thus :

$$\frac{d}{dx}\phi_n(x)=np_0x^{n-1}+(n-1)np_1x^{n-2}+(n-2)\frac{n(n-1)}{2}p_2x^{n-3}$$

$$+\ldots\ldots+np_{n-1},$$

$$=n\{p_0x^{n-1}+(n-1)p_1x^{n-2}+\frac{(n-1)(n-2)}{2}p_2x^{n-3}$$

$$+\ldots\ldots+p_{n-1}\},$$

$$=n\phi_{n-1}(x)\;;$$

$$\frac{d^2}{dx^2}\phi_n(x)=n\frac{d}{dx}\phi_{n-1}(x)=n(n-1)\phi_{n-2}(x)\;;$$

....

$$\frac{d^r}{dx^r}\phi_n(x)={}^np_r\phi_{n-r}(x)\;;$$

....

and $\dfrac{d^n}{dx^n}\phi_n(x)=n\,!\;\phi_0(x)=n\,!\;p_0.$

Hence

$$\phi_n(x+h)=\sum_{r=0}^{n}\frac{\phi_n^{(r)}(h)}{r\,!}x^r=\sum_{r=0}^{n}\frac{{}^n\mathrm{P}_r}{r\,!}\phi_{n-r}(h)x^r,$$

$$=\sum_{r=0}^{n}{}^n\mathrm{C}_{n-r}\phi_{n-r}(h)x^r=\sum_{r=0}^{n}{}^n\mathrm{C}_r\phi_r(h)x^{n-r}.$$

$\phi_n(x+h)$ is thus obtained by writing $\phi_r(h)$ for p_r in the polynomial $\phi_n(x)$.

Example.

$$\begin{aligned}\phi_3(x+h)&=\phi_0(h)x^3+3\phi_1(h)x^2+3\phi_2(h)x+\phi_3(h),\\&=p_0x^3+3(p_0h+p_1)x^2+3(p_0h^2+2p_1h+p_2)x\\&\qquad+(p_0h^3+3p_1h^2+3p_2h+p_3).\end{aligned}$$

CHAPTER II

Relations between the Roots and Co-efficients of Equations.

§ **13.** If $a_1, a_2, a_3 \ldots\ldots a_n$ be the n roots of the equation
$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0\,;$$
then (as in § 9), we have
$$p_0x^n+p_1x^{n-1}+p_2x^{-2}+\ldots+p_n$$
$$\equiv p_0(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n).$$
Dividing both sides of this identity by p_0, we get
$$x_n+\frac{p_1}{p_0}x^{n-1}+\frac{p_2}{p_0}x^{n-2}+\ldots\ldots+\frac{p_n}{p_0}$$
$$\equiv(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n).$$

The right hand side of this identity is the product of n factors of the type $(x-a_r)$. Every term in the product would be formed by multiplying together n letters, one taken from each of the n factors. Thus, each term involving x^{n-r} would be obtained by taking x out of any $(n-r)$ factors and $(-a)$'s out of the remaining r factors. The co-efficient of x^{n-r} in the product would, therefore, be the sum of the products of the n quantities : $-a_1$, $-a_2$, $-a_3\ldots\ldots$, $-a_n$ taken r at a time ; *i. e.* $(-1)^r$ S_r where S_r stands for the sum of the products of the n roots $a_1, a_2, a_3\ldots\ldots\ldots, a_n$ taken r at a time.

Equating the co-efficients of like powers of x on the two sides of the identity, we obtain the following relations between the roots and co-efficients of the given equation :—

$$\frac{p_1}{p_0}=-S_1=-(a_1+a_2+a_3+\ldots+a_n),$$

$$\frac{p_2}{p_0}=S_2=(a_1a_2+a_1a_3+a_1a_4+\ldots+a_{n-1}a_n),$$

$$\frac{p_3}{p_0}=-S_3=-(a_1a_2a_3+a_1a_2a_4+a_1a_2a_5+\ldots\ldots+a_{n-2}a_{n-1}a_n),$$

$$\frac{p_r}{p_0}=(-1)^r S_r.$$

....

and $\dfrac{p_n}{p_0}=(-1)^n S_n=(-1)^n a_1 a_2 a_3 \ldots\ldots a_n.$

In particular, when $p_0=1$ we obtain

$$S_1=-p_1,\ S_2=p_2,\ S_3=-p_3, \ldots\ldots, S_r=(-1)^r p_r \ldots\ldots$$

and $$S_n=(-1)^n p_n.$$

§14. In the case of the quadratic equation :

$$ax^2+bx+c=0,$$

if α, β be the two roots, the relations of the preceding article take the well known form :

$$\alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}.$$

In the case of the cubic

$$ax^3+bx^2+cx+d=0,$$

if α, β, γ denote the three roots, then we have

$$\boxed{\alpha+\beta+\gamma=-\frac{b}{a},\ \alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a} \text{ and } \alpha\beta\gamma=-\frac{d}{a}.}$$

If α, β, γ, δ be the four roots of the biquadratic :

$$ax^4+bx^3+cx^2+dx+e=0,$$

we obtain the relations :

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a},$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a},$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a},$$

and $$\alpha\beta\gamma\delta=\frac{e}{a}.$$

Example 1. Solve the equation $x^3-5x^2-16x+80=0$, the sum of two of its roots being zero.

Let $\alpha, -\alpha, \beta$ be the three roots of the given cubic, then

$$S_1 = \alpha - \alpha + \beta = 5. \qquad (i)$$
$$S_2 = -\alpha^2 + \alpha\beta - \alpha\beta = -16, \qquad (ii)$$
and
$$S_3 = -\alpha^2\beta = -80. \qquad (iii)$$

From (i), we get $\beta = 5$.

From (ii), we obtain $\alpha = 4$ or -4.

These values satisfy also the last of the three relations.

Thus 4, -4, 5 are the three roots of the given equation.

Example 2. Solve the equation $x^3 - 12x + 16 = 0$, two of its roots being equal.

Let α, α, β be the three roots of the given cubic, then

$$S_1 = 2\alpha + \beta = 0,$$
$$S_2 = \alpha^2 + \alpha\beta + \alpha\beta = -12,$$
and
$$S_3 = \alpha^2\beta = -16.$$

Mark what is the process. Apply Rolle's theorem

The first of the three relations gives $\beta = -2\alpha$.

Substituting in the last two, we get $\alpha^2 = 4$ and $\alpha^3 = 8$.

Therefore $\alpha = 2$, and $\beta = -4$.

Thus the required roots are 2, 2, -4.

Example 3. Find the conditions that the roots of the equation $x^3 - px^2 + qx - r = 0$, may be (i) in A. P., (ii) in G.P., and (iii) in H.P.

(i) Let $\alpha - \delta, \alpha, \alpha + \delta$ be the three roots.

Then
$$S_1 = 3\alpha = p,\ S_2 = (\alpha - \delta)\alpha + (\alpha^2 - \delta^2) + \alpha(\alpha + \delta) = q$$
$$\text{and } S_3 = \alpha(\alpha^2 - \delta^2) = r.$$

We have to eliminate α and δ between these three relations, which when simplified take the form :

$$\alpha = \frac{p}{3}\ ;\ 3\alpha^2 - \delta^2 = q\ ;\ \alpha^2 - \delta^2 = \frac{r}{\alpha}.$$

Subtracting the last from the second of these trelaions :

we get $2\alpha^2 = q - \dfrac{r}{\alpha}$.

Substituting for α from the first, we get

$$\frac{2p^2}{9}=q-\frac{3r}{p}, \text{ i.e. } 2p^3-9pq+27r=0.$$

(*ii*) Let $\frac{\alpha}{\rho}$, α, $\alpha\rho$ be the three roots of the given equation.

Then, we have

$$S_1=\frac{\alpha}{\rho}+\alpha+\alpha\rho=p, \quad \text{i.e. } \frac{\alpha}{\rho}(1+\rho+\rho^2)=p\,; \qquad (i)$$

$$S2=\frac{\alpha^2}{\rho}+\alpha^2+\alpha^2\rho=q, \text{ i. e. } \frac{\alpha^2}{\rho}(1+\rho+\rho^2)=q\,; \qquad (ii)$$

and $S_3=\alpha^3=r$. (*iii*)

Dividing (*ii*) by (*i*), we obtain $\alpha=\frac{q}{p}$.

Substituting $\frac{q}{p}$ for α in (*iii*), the required condition is

$$\frac{q^3}{p^3}=r, \text{ i.e. } q^3=p^3r.$$

(*iii*) Let $\frac{1}{\alpha-\delta}$, $\frac{1}{\alpha}$, $\frac{1}{\alpha+\delta}$ be the three roots.

Then $S_1=\frac{1}{\alpha-\delta}+\frac{1}{\alpha}+\frac{1}{\alpha+\delta}=p,$

$$S_2=\frac{1}{(\alpha-\delta)\alpha}+\frac{1}{\alpha^2-\delta^2}+\frac{1}{\alpha(\alpha+\delta)}=q,$$

and $$S_3=\frac{1}{\alpha(\alpha^2-\delta^2)}=r.$$

Simplifying, we get

$$\alpha(\alpha^2-\delta^2)=\frac{1}{r}\,;\ \frac{3\alpha}{\alpha(\alpha^2-\delta^2)}=q\,;\ \frac{3\alpha^2-\delta^2}{\alpha(\alpha^2-\delta^2)}=p.$$

Eliminating α and δ from these three equations, we get the required condition *viz.* $2q^3-9pqr+27r^2=0$.

Example 4. Solve the equation :

$$x^3-9x^2+14x+24=0,$$

two of the roots being in the ratio of 3 : 2.

Let 3α, 2α, β be the three roots.

Then $$S_1=5\alpha+\beta=9, \quad (i)$$

$$S_2=6\alpha^2+5\alpha\beta=14, \quad (ii)$$

and $$S_3=6\alpha^2\beta=-24. \quad (iii)$$

From (*i*), $\beta=9-5\alpha$.

Substituting in (*ii*), we get

$$19\alpha^2-45\alpha+14=0,$$

whence $\alpha=2$ or $\frac{7}{19}$, $\beta=-1$ or $7\frac{3}{19}$.

The values $\frac{7}{19}$ and $7\frac{3}{19}$ of α and β, do not satisfy the last of the above relations, and have therefore to be rejected. Thus the roots are 6, 4, -1.

Example 5. Solve the equations :

$$x+ay+a^2z=a^3,$$
$$x+by+b^2z=b^3,$$
$$x+cy+c^2z=c^3,$$

for x, y, z.

From the given equations, it is readily seen that a, b, c are the roots of the cubic,

$$t^3=zt^2+yt+x\ ;$$

i. e. $$t^3-zt^2-yt-x=0.$$

Hence $z=a+b+c$; $y=-(ab+bc+ca)$; $x=abc$.

Example 6. If $1, \alpha, \beta, \gamma$......be the roots of the equation

$$x^n-1=0,$$

prove that $(1-\alpha)(1-\beta)(1-\gamma)......=n$.

We have $x^n-1\equiv(x-1)[(x-\alpha)(x-\beta)(x-\gamma)...]$.

Differentiating both sides of this identity with respect to x, we get

$$nx^{n-1}\equiv(x-a)(x-\beta)(x-\gamma)...+(x-1)\ \frac{d}{dx}[(x-\alpha)(x-\beta)...].$$

$\alpha, \alpha\beta.$

Writing unity for x in this identity, we have

$$n=(1-\alpha)(1-\beta)(1-\gamma)\ldots\ldots$$

EXERCISES—X

Solve the equations :

1. $4x^3+16x^2-9x-36=0$, the sum of two of the roots being zero.

2. $4x^3+20x^2-23x+6=0$, two of its roots being equal.

3. $3x^3-26x^2+52x-24=0$, the roots being in G.P.

4. $32x^3-48x^2+22x-3=0$, the roots being in A.P.

5. $40x^4+22x^3-21x^2-2x+1=0$, the roots being in H.P.

6. $2x^3-x^2-22x-24=0$, two of the roots being in the ratio of 3 : 4.

7. $8x^4-2x^3-27x^2+6x+9=0$, two of its roots being equal but opposite in sign.

8. $6x^4-29x^3+40x^2-7x-12=0$, the product of two of its roots being 2.

9. $27x^4-195x^3+494x^2-520x+192=0$, the roots being in G.P.

[Ans. 1. $-4, \pm 3/2$. 2. $\frac{1}{2}, \frac{1}{2}, -6$. 3. $6, 2, 2/3$.
4. $1/4, 1/2, 3/4$. 5. $-\frac{1}{4}, -1, \frac{1}{2}, \frac{1}{5}$.
6. $-3/2, -2, 4$. 7. $\pm\sqrt{3}, 3/4, -1/2$.
8. $3/2, 4/3, 1\pm\sqrt{2}$. 9. $8/9, 4/3, 2, 3$.]

10. Find the conditions that the equation $x^3-px^2+qx-r=0$ may have (*i*) two equal roots, (*ii*) two roots equal but opposite in sign, (*iii*) all the three roots equal.

[Ans. (*i*) $(9r-pq)^2=4(q^2-3pr)(p^2-3q)$.
(*ii*) $pq=r$. (*iii*) $p^2=3q$, $p^3=27r$.]

11. Find the conditions that the equation :

$$x^4+px^3+qx^2+rx+s=0,$$

may have (*i*) its roots in A.P., (*ii*) its roots in G.P., (*iii*) its roots all equal, (*iv*) three roots equal, (*v*) two pairs of equal roots, (*vi*) two roots equal but opposite in sign.

[**Solution.** (*i*) Let $\alpha-3\delta$, $\alpha-\delta$, $\alpha+\delta$, $\alpha+3\delta$ be the four roots in question. Then, we have

$$4\alpha=-p,\ 6\alpha^2-10\delta^2=q,\ 4\alpha(\alpha^2-5\delta^2)=-r$$

and $(\alpha^2-\delta^2)(\alpha^2-9\delta^2)=s$.

Eliminating α and δ between these relations, we obtain the two necessary conditions *viz.*, $p^3-4pq+8r=0$

and $(36q-11p^2)(4q+p^2)=1600s$.

(*ii*) Let $\frac{\alpha}{\rho^3}$, $\frac{\alpha}{\rho}$, $\alpha\rho$, $\alpha\rho^3$ be the roots in question. Then

$$\alpha\left(\frac{1}{\rho^3}+\frac{1}{\rho}+\rho+\rho^3\right)=-p, \qquad (a)$$

$$\alpha^2\left(\frac{1}{\rho^4}+\frac{1}{\rho^2}+2+\rho^2+\rho^4\right)=q, \qquad (b)$$

$$\alpha^3\left(\frac{1}{\rho^3}+\frac{1}{\rho}+\rho+\rho^3\right)=-r, \qquad (c)$$

and

$$\alpha^4=s. \qquad (d)$$

Eliminating α and ρ between (*a*), (*c*) and (*d*), we get one of the conditions *viz.*, $r^2=p^2s$.

The other condition is obtained by eliminating α and ρ between (*a*), (*b*) and (*c*). For this purpose, we put

$$\rho+\frac{1}{\rho}=t, \text{ so that } \rho^2+\frac{1}{\rho^2}=t^2-2,$$

$$\rho^3+\frac{1}{\rho^3}=t^3-3t \text{ and } \rho^4+\frac{1}{\rho^4}=t^4-4t^2+2.$$

We then have

$$t^3-2t+\frac{p}{\alpha}=0, \qquad (e)$$

and

$$t^4-3t^2+2=\frac{q}{\alpha^2}=\frac{pq}{r}. \qquad (f)$$

From (*e*) and (*f*), we get

$$t^2+\frac{p}{\alpha}t+\left(\frac{pq}{r}-2\right)=0. \qquad (g)$$

From (*e*) and (*g*), we obtain

$$\frac{p}{\alpha}t^2+\frac{pq}{r}t-\frac{p}{\alpha}=0,$$

or
$$\frac{t^2}{\alpha}+\frac{qt}{r}-\frac{1}{\alpha}=0. \qquad (h)$$

From (g) and (h), we have

$$\frac{t^2}{-\frac{p}{\alpha^2}+\frac{2q}{r}-\frac{pq^2}{r^2}}=\frac{t}{-\frac{2}{\alpha}+\frac{pq}{r\alpha}+\frac{1}{\alpha}}=\frac{1}{\frac{q}{r}-\frac{p}{\alpha^2}}$$

Hence

$$\frac{1}{r^2\alpha^2}(pq-r)^2=\left(\frac{q}{r}-\frac{p}{\alpha^2}\right)\left(\frac{2q}{r}-\frac{pq^2}{r^2}-\frac{p}{\alpha^2}\right);$$

i.e.,
$$\frac{p}{r^3}(pq-r)^2=\left(\frac{q}{r}-\frac{p^2}{r}\right)\left(\frac{2q}{r}-\frac{pq^2}{r^2}-\frac{p^2}{r}\right);$$

or $p(pq-r)^2=(p^2-q)(pq^2+p^2r-2qr)$;

or which is the same thing,

$$(q^2-pr)(pq-r)=r(p^2-q)^2.$$

This is the second necessary condition.

(iii) The necessary conditions are :

(a) $8q=3p^2$, (b) $16r=p^3$. (c) $256s=p^4$.

(iv) The necessary conditions are :

(a) $(4q^2-9pr)(3p^2-8q)=3(6r-pq)^2$,

(b) $(pqr+8qs-9p^2s)(pr-16s)=3(6ps-rq)^2$.

(v) Let α, α, β, β denote the roots in question. Then,

$$2(\alpha+\beta)=-p,$$
$$\alpha^2+\beta^2+4\alpha\beta=q,$$
$$2\alpha\beta(\alpha+\beta)=-r,$$

and
$$\alpha^2\beta^2=s.$$

Eliminating α, β between every three of these relations, we obtain the following four conditions :

(a) $p^3+8r=4pq$, (b) $p^2s=r^2$,

(c) $(4q-p^2)^2=64s$, (d) $(r^2-4qs)^2=64s^3$.

(*vi*) $p^2s+r^2-pqr=0$].

§**15.** *If the equation $f_n(x)=0$ has r of its roots each equal to a, then the equation $f'_n(x)=0$ has $(r-1)$ of its roots each equal to a.*

Let $f_{n-r}(x)$ denote the quotient when $f_n(x)$ is divided by $(x-a)^r$; then $f_n(x)\equiv(x-a)^r f_{n-r}(x)$.

Differentiating,

$$\begin{aligned} f'_n(x) &= r(x-a)^{r-1}f_{n-r}(x)+(x-a)^r f'_{n-r}(x), \\ &= (x-a)^{r-1}[rf_{n-r}(x)+(x-a)f'_{n-r}(x)]\,; \end{aligned}$$

so that $f'_n(x)$ contains the factor $(x-a)^{r-1}$.

Hence the equation $f'_n(x)=0$ has $(r-1)$ of its roots each equal to a.

Ex. Is the converse true ?

Ex. Show that if $f_n(x)=0$ has only r of its roots each equal to a, then $f'_n(x)=0$ has only $(r-1)$ of its roots each equal to a.

§**16.** From the preceding article, it readily follows that an equation $f(x)=0$ does or does not have equal roots according as $f(x)$ and $f'(x)$ do or do not have a common factor involving x. Moreover, if $f(x)=0$ has r of its roots each equal to a, then $f^{(r-1)}(x)=0$ has one of its roots equal to a. Hence, if $f(x)$ and $f^{(r-1)}(x)$ have a common factor $(x-a)$, then $f(x)=0$ has r of its roots each equal to a. The common roots of two equations $F(x)=0$ and $G(x)=0$ will be obtained by equating to zero the H.C.F. of $F(x)$ and $G(x)$.

Example 1. Solve the equation :

$$6x^5-25x^4+41x^3-33x^2+13x-2=0,$$

which has three equal roots.

Let $f(x)\equiv6x^5-25x^4+41x^3-33x^2+13x-2$,

then $f'(x)\equiv30x^4-100x^3+123x^2-66x+13$,

and $f''(x)\equiv120x^3-300x^2+246x-66$.

Since $f(x)=0$ has three equal roots ; $f'(x)$ and $f''(x)$ have a common factor involving x.

This common factor is obtained as follows :

$$
\begin{array}{r|r|r|l}
 & 6)120-300+246-66 & 30-100+123-66+13 & \\
5 & 20-50+41-11 & 2 & \\
 & 20-35+15 & 60-200+246-132+26 & 3 \\
 & -15+26-11 & 60-150+123-33 & \\
 & 4 & -50+123-99+26 & \\
-15 & -60+104-44 & 2 & \\
 & -60+105-45 & -100+246-198+52 & -5 \\
 & -1)\ -1\ +1 & -100+250-205+55 & \\
 & 1\ -1 & -1\ )-4+7-3 & \\
 & & 4-7+3 & 4 \\
 & & 4-4 & \\
 & & -3+3 & -3 \\
 & & -3+3 & \\
 & & \times &
\end{array}
$$

Thus the required common factor is $(x-1)$.

Since $f(x)=0$ has three equal roots, $(x-1)^3$ must occur as a factor in $f(x)$. The remaining factor is obtained by dividing $f(x)$ by $(x-1)$ three times in succession as follows :

$$
\begin{array}{r|l}
 & 6-25+41-33+13-2 \\
1 & \quad +6-19+22-11+2 \\
\hline
 & 6-19+22-11+2\ |+0 \\
1 & \quad +6-13+9-2 \\
\hline
 & 6-13+9-2\ |+0 \\
1 & \quad +6-7+2 \\
\hline
 & 6-7+2\ |+0
\end{array}
$$

The quadratic equation :

$$6x^2-7x+2=0.$$

has $\frac{1}{2}$ and $\frac{2}{3}$ as its roots. Hence the roots of the given equation are : 1, 1, 1, $\frac{1}{2}$ and $\frac{2}{3}$.

N.B.—Instead of dividing $f(x)$ by $(x-1)$ thrice in succession, we could divide it by x^3-3x^2+3x-1, by the method of synthetic division given in Chapter I. The process would then be exhibited thus :

$$\begin{array}{r|llllll}
 & 6 & -25 & +41 & -33 & +13 & -2 \\
3 & & +18 & -18 & +6 & & \\
-3 & & & -21 & +21 & -7 & \\
1 & & & & +6 & -6 & +2 \\
\hline
1 & 6 & -7 & +2 \mid & +0 & +0 & +0
\end{array}$$

Example 2. Solve the equations :

$$f(x) \equiv x^4 + 5x^3 - 22x^2 - 50x + 132 = 0,$$

$$\text{and } \phi(x) \equiv x^4 + x^3 - 20x^2 + 16x + 24 = 0\ ;$$

which have some of their roots in common.

The common roots will be obtained by equating to zero the H. C. F. of the expressions $f(x)$ and $\phi(x)$. This is obtained as follows :—

$$\begin{array}{r|r|r|l}
 & 1+\ \ 1-20+16+\ 24 & 1+5-22-50+132 & 1 \\
 & 2 & 1+1-20+16+\ 24 & \\
\cline{2-3}
1 & 2+\ \ 2-40+32+\ 48 & 2\ )\ 4-\ 2-66+108 & \\
 & 2-\ \ 1-33+54 & 2-\ 1-33+\ 54 & 2 \\
\cline{2-2}
 & 3-\ 7-22+\ 48 & 2-10+12 & \\
\cline{3-3}
 & 2 & 9-45+\ 54 & 9 \\
\cline{2-2}
3 & 6-14-44+\ 96 & 9-45+\ 54 & \\
\cline{3-3}
 & 6-\ 3-99+162 & \times & \\
\cline{2-2}
 & -11)-11+55-\ 66 & & \\
\cline{2-2}
 & 1-\ 5+\ \ 6 & &
\end{array}$$

By equating to zero the common factor $x^2 - 5x + 6$, we obtain 2 and 3 as the common roots of the two equations $f(x) = 0$ and $\phi(x) = 0$.

Dividing $f(x)$ and $\phi(x)$ by $x^2 - 5x + 6$ as follows :

$$\begin{array}{r|lllll}
 & 1 & +5 & -22 & -50 & +132 \\
5 & & +5 & -6 & & \\
-6 & & & +50 & -60 & \\
 & & & & +110 & -132 \\
\hline
1 & 1 & +10 & +22 \mid & +0 & +0
\end{array}
\qquad
\begin{array}{r|lllll}
 & 1 & +1 & -20 & +16 & +24 \\
5 & & +5 & -6 & & \\
-6 & & & +30 & -36 & \\
 & & & & +20 & -24 \\
\hline
1 & 1 & +6 & +4 \mid & +0 & +0
\end{array}$$

we see that the remaining roots of $f(x) = 0$ and $\phi(x) = 0$ are given by the quadratics :

$$x^2 + 10x + 22 = 0 \text{ and } x^2 + 6x + 4 = 0 \text{ respectively.}$$

Thus the roots of the equation $f(x)=0$ are $2, 3, -5\pm\sqrt{3}$ and those of $\phi(x)=0$ are $2, 3, -3\pm\sqrt{5}$.

Example 3. The equation

$$f(x)\equiv x^4+15x^3+70x^2+120x+64=0,$$

has two roots which differ by 3. Solve the equation.

Let α and $\alpha-3$ be the roots in question.

Then $f(\alpha)=0$ and $f(\alpha-3)=0$.

We find $f(\alpha-3)$ as follows :

$$\begin{array}{r|l}
 & 1+15+70+120+64 \\
-3 & \underline{\quad -\ 3-36-102-54} \\
 & 1+12+34+\ 18 \mid +10 \\
-3 & \underline{\quad -\ 3-27-\ 21} \\
 & 1+\ 9+\ 7 \mid -3 \\
-3 & \underline{\quad -\ 3-18} \\
 & 1+\ 6 \mid -11 \\
-3 & \underline{\quad -\ 3} \\
 & 1+\ 3
\end{array}$$

The required value of α is the common root of the equations :

$$\alpha^4+15\alpha^3+70\alpha^2+120\alpha+64=0,\ \alpha^4+3\alpha^3-11\alpha^2-3\alpha+10=0.$$

This is obtained by equating to zero the H. C. F. of $f(\alpha)$ and $f(\alpha-3)$.

The H. C. F. is obtained as follows :

$$\begin{array}{r|r|r|l}
 & 1+\ 3-\ 11-\ \ 3+\ 10 & 1+15+\ 70+\ 120+\ \ 64 & 1 \\
 & \underline{4\qquad\qquad\qquad\qquad} & \underline{1+\ 3-\ 11-\qquad 3+\ \ 10} & \\
1 & 4+12-\ 44-\ 12+\ 40 & 3)\ 12+\ 81+\ 123+\ \ 54 & \\
 & \underline{4+27+\ 41+\ 18\qquad} & 4+\ 27+\ \ 41+\ \ 18 & \\
 & -15-\ 85-\ 30+\ 40 & \underline{13\qquad\qquad\qquad\qquad} & \\
 & \underline{4\qquad\qquad\qquad} & 52+351+\ 533+\ 234 & 4 \\
-15 & -60-340-120+160 & \underline{52+396+\ 344\qquad} & \\
 & \underline{-60-405-615-270} & -45+\ 189+\ 234 & \\
13 & 5)\ \ 65+495+430 & \underline{13\qquad\qquad\qquad} & \\
 & 13+\ 99+\ 86 & -585+2457+3042 & -45 \\
 & \underline{13+\ 13\qquad} & \underline{-585-4455-3870} & \\
86 & 86+\ 86 & 6912)6912+6912 & \\
 & \underline{86+\ 86} & 1+1 & \\
 & \times & &
\end{array}$$

The H.C.F. is $\alpha+1$. Therefore $\alpha=-1$.

Hence -1 and -4 are the two roots of $f(x)=0$ which differ by 3.

Dividing $f(x)$ by $(x+1)$ and $(x+4)$ in succession or by x^2+5x+4 by the method of synthetic division :

$$
\begin{array}{r|l}
 & 1+15+70+120+64 \\
-5 & \quad -5-\;4 \\
-4 & \qquad -50-\;40 \\
 & \qquad\qquad -\;80-64 \\ \hline
1 & 1+10+16+\;\;0+\;0
\end{array}
$$

We get the quadratic

$$x^2+10x+16=0.$$

This gives the remaining roots *viz.*, -2 and -8 of $f(x)=0$.

Example 4. Solve the equation :

$$f(x)\equiv x^3-5x^2-16x+80=0$$

the sum of two of its roots being zero.

Let α and $-\alpha$ be the roots in question.

Then $f(\alpha)=\alpha^3-5\alpha^2-16\alpha+80=0,$ (*i*)

and $f(-\alpha)=-\alpha^3-5\alpha^2+16\alpha+80=0.$ (*ii*)

The value of α is found by equating to zero the H.C.F. of $f(\alpha)$ and $f(-\alpha)$. This is found to be α^2-16, so that $\alpha=4$ or -4.

Therefore, the roots of $f(x)=0$ in question are 4 and -4.

$$
\begin{array}{r|r|r}
1 & 1-5-16+80 & 1+5-16-\;80 \\
 & 1+0-16 & 1-5-16+\;80 \\ \hline
-5 & -5+\;0+80 & 10\,)10+\;0-160 \\
 & -5+\;0+80 & 1+\;0-\;16 \\ \hline
 & \times &
\end{array}
$$

The remaining root is obtained by equating to zero the quotient of $f(x)$ by (x^2-16).

$$
\begin{array}{r|l}
 & 1-5-16+80 \\
0 & \quad +0+16 \\
16 & \qquad +\;0-80 \\ \hline
1 & 1-5\;|\;+0+0
\end{array}
$$

We thus have

$$x-5=0.$$

Therefore, 5 is the remaining root of the given equation.

Example 5. Find the condition that the cubic

$$\phi(x) \equiv ax^3+3bx^2+3cx+d=0.$$

may have two equal roots.

If $\phi(x)=0$ has two equal roots, the equations :

$$\phi(x) \equiv ax^3+3bx^2+3cx+d=0, \qquad (i)$$

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$$\text{and } \phi'(x) \equiv 3(ax^2+2bx+c)=0, \qquad (ii)$$

must have a common root. We have to eliminate x from (i) and (ii).

Multiply (ii) by $x/3$ and subtracting from (i), we have

$$bx^2+2cx+d=0. \qquad (iii)$$

From (ii) and (iii) we get

$$\frac{x^2}{2(bd-c^2)}=\frac{x}{bc-ad}=\frac{1}{2(ac-b^2)}.$$

Therefore, the required condition is

$$(bc-ad)^2=4(ac-b^2)(bd-c^2).$$

Example 6. The equation

$$x^5-209x+56=0,$$

has two roots whose product is unity. Determine them.

(I. A. and A. S., Allahabad, 1926)

Let α and $\frac{1}{\alpha}$ be the two roots in question. Then

$$\alpha^5-209\alpha+56=0, \qquad (i)$$

and $$\frac{1}{\alpha^5}-\frac{209}{\alpha}+56=0,$$

i.e., $$56\alpha^5-209a^4+1=0. \qquad (ii)$$

$$
\begin{array}{c|r|r|l}
1 & 1+0+0+0-209+56 & 56-209+0+0+0+1 & 56 \\
 & 1+0-0-56+15 & 56+0+0+0-11704+3136 & \\
\hline
 & 56)\,56-224+56 & -209)\,-209+0+0+11704-3135 & \\
\hline
 & 1-4+1 & 1+0+0-56+15 & 1 \\
 & & 1-4+1 & \\
\hline
 & & 4-1-56+15 & 4 \\
 & & 4-16+4 & \\
\hline
 & & 15-60+15 & 15 \\
 & & 15-60+15 & \\
\hline
 & & \times &
\end{array}
$$

Equating to zero the H. C. F. $\alpha^2-4\alpha+1$ of the expressions $\alpha^5-209\alpha+56$ and $56\alpha^5-209\alpha^4+1$,

we get $\alpha=2\pm\sqrt{3}$, so that $\dfrac{1}{\alpha}=2\mp\sqrt{3}$.

The two roots in question are thus $2+\sqrt{3}$ and $2-\sqrt{3}$.

§ **17**. *If a relation between two roots of an equation be given, the degree of an equation can be reduced by two.*

Let $f(x)=0$ be an equation of the nth degree.

Also let $\alpha=\phi(\beta)$ be the relation between α and β, two of the roots of the equation $f(x)=0$, ϕ being an algebraic function.

Since α and β are roots of $f(x)=0$,

$$f(\alpha)=0 \text{ and } f(\beta)=0\,;$$

$$\textit{i.e.,}\ f\{\phi(\beta)\}=0 \text{ and } f(\beta)=0.$$

Equating to zero the H. C. F. of $f(\beta)$ and $f\{\phi(\beta)\}$, we shall get β. The value of α is now given by the relation $\alpha=\phi(\beta)$. Two of the roots of $f(x)=0$ being thus known, the degree of the equation can be lowered by two.

EXERCISES XI

1. The equation $8x^4+4x^3-18x^2+11x-2=0$, has three equal roots. Solve it. **Ans.** $^1/_2$, $^1/_2$, $^1/_2$, -2.

2. The following equations have some equal roots. Solve them.

(a) $x^5-x^3+4x^2-3x+2=0$. Ans. $\frac{1\pm i\sqrt{3}}{2}$, $\frac{1\pm i\sqrt{3}}{2}$, -2.

(b) $x^6-2x^5-4x^4+12x^3-3x^2-18x+18=0$.
Ans. $\pm\sqrt{3}$, $\pm\sqrt{3}$, $1\pm i$.

(c) $x^5-13x^4+67x^3-171x^2+216x-108=0$.
Ans. 3, 3, 3, 2, 2.

(d) $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$.
Ans. 2, 2, 1/2, 1/2, $\frac{1\pm i\sqrt{3}}{2}$.

3. Find the condition that the equation $x^n-px^2+r=0$ may have equal roots. Ans. $n^n r^{n-2}=4p^n(n-2)^{n-2}$.

4. If the equation $x^4+ax^3+bx^2+cx+d=0$, has three equal roots show that each of them is equal to $(6c-ab)/(3a^2-8b)$. Hence solve the equation :

$$8x^4+4x^3-126x^2+243x-135=0.$$

Ans. $\frac{3}{2}$, $\frac{3}{2}$, $\frac{3}{2}$, -5.

5. The equations $4x^4+12x^3-x^2-15x=0$, and $6x^4+13x^3-4x^2-15x=0$, have some of their roots common. Solve them.
Ans. 0, 1, $-3/2$, $-5/2$; 0, 1, $-3/2$, $-5/3$.

6. Show that the equation $x^4+px^2+q=0$, cannot have three equal roots.

7. Find the value of a/b so that the equations :

$$ax^2+bx+a=0 \text{ and } x^3-2x^2+2x-1=0.$$

may have (*i*) one root in common, and (*ii*) two roots in common.
Ans. (*i*) $-1/2$; (*ii*) -1.

8. If the equation $x^5+qx^3+rx^2+t=0$ has two equal roots, prove that one of them will be a root of the quadratic

$$15rx^2-6q^2x+25t-4qr=0.$$

9. Show that the equation $\sum_{r=0}^{n} {}^nP_r \; x^{n-r}=0$ cannot have equal roots.

10. If the equation $x^5-10a^3x^2+b^4x+c^5=0$ has three equal roots, show that $ab^4-9a^5+c^5=0$.

11. The equation $x^5-409x+285=0$ has two roots whose sum is 5. Determine them. **Ans.** $\frac{5\pm\sqrt{13}}{2}$.

12. If the in equation $x^4-px^3+qx^2-rx+s=0$, the sum of two of the roots be equal to the sum of the other two, prove that $p^3-4pq+8r=0$.
If in the same equation the product of two of the roots be equal to the product of the other two, show that $r^2=p^2s$.

13. Solve : $x^3+x^2-16x+20=0$, the difference between two of the roots being 7. **Ans.** $-5, 2, 2$.

14. The equation $800\,x^4-102x^2-x+3=0$ has two equal roots. Solve it. **Ans.** $-1/4, -1/4, 1/5.\ 3/10$.

CHAPTER III

Symmetric Functions of the Roots of an Equation

§18. Symmetric functions of the roots of an equation are those functions that remain unaltered in value when any two of the roots are interchanged. We had examples of such functions in the preceding chapter, while considering the relations between the roots and co-efficients of an equation. Thus s_1, s_2, s_3.........s_n were all symmetric functions of the roots, for the roots were symmetrically involved in each of them. If one term of a symmetric function be given, the others can be put down readily by taking different permutations of the roots. It is, therefore, customary to represent a symmetric function of the roots by placing a sigma (Σ) before one of the terms of the function. For example, if

α, β, γ be the roots of a cubic, $\Sigma\alpha^3\beta$ represents the symmetric function : $\alpha^3\beta+\alpha^3\gamma+\beta^3\alpha+\beta^3\gamma+\gamma^3\alpha+\gamma^3\beta$, and if α, β, γ, δ be the roots of a biquadratic, the same symbol represents

$$\alpha^3(\beta+\gamma+\delta)+\beta^3(\gamma+\alpha+\delta)+\gamma^3(\alpha+\beta+\delta)+\delta^3(\alpha+\beta+\gamma).$$

The value of any symmetric function can be found in terms of the co-efficients with the help of the relations of the preceding chapter.

Example 1. If α, β, γ, δ be the roots of the biquadratic

$$x^4+px^3+qx^2+rx+s=0,$$

find in terms of the co-efficients the value of

$$(i)\ \Sigma\alpha^2,\ (ii)\ \Sigma(\beta+\gamma+\delta)^2.$$

We have $\alpha+\beta+\gamma+\delta=-p,$

and $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q.$

Therefore, $\Sigma\alpha^2=(\alpha^2+\beta^2+\gamma^2+\delta^2)$

$$=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

$$=p^2-2q\ ;$$

Also $\Sigma(\beta+\gamma+\delta)^2=\Sigma[(\alpha+\beta+\gamma+\delta)-\alpha]^2$
$=\Sigma(-p-\alpha)^2=\Sigma(p+\alpha)^2$
$=(p+\alpha)^2+(p+\beta)^2+(p+\gamma)^2+(p+\delta)^2.$
$=4p^2+2p\Sigma\alpha+\Sigma\alpha^2$
$=4p^2-2p^2+p^2-2q=3p^2-2q.$

Example 2. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic

$$p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0,$$

find the value of :

$$(3\alpha-\beta-\gamma-\delta)(3\beta-\gamma-\delta-\alpha)(3\gamma-\delta-\alpha-\beta)(3\delta-\alpha-\beta-\gamma).$$

We have $\alpha+\beta+\gamma+\delta=-4p_1/p_0$.

Therefore, $3\alpha-\beta-\gamma-\delta=4\left(\alpha+\dfrac{p_1}{p_0}\right).$

Moreover $p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4$
$\equiv p_0(x-\alpha)(x-\beta)(x-\gamma)(x-\delta),$

Substituting $-\dfrac{p_1}{p_0}$ for x in this identity, we obtain

$$p_0(\alpha+\frac{p_1}{p_0})(\beta+\frac{p_1}{p_0})(\gamma+\frac{p_1}{p_0})(\delta+\frac{p_1}{p_0})$$

$$=p_0(-\frac{p_1}{p_0})^4+4p_1(-\frac{p_1}{p_0})^3+6(-\frac{p_1}{p_0})^2p_2+4p_3(-\frac{p_1}{p_0})+p_4,$$

$$=\frac{1}{p_0^3}\left\{p_0^3p_4-4p_0^2p_1p_3+6p_0p_1^2p_2-3p_1^4\right\}.$$

Hence, $\Pi(3\alpha-\beta-\gamma-\delta)=\Pi[4(\alpha+\frac{p_1}{p_0})]=256\ \Pi(\alpha+\frac{p_1}{p_0})$

$$=\frac{256}{p_0^4}[p_0^3p_4-4p_0^2p_1p_3+6p_0p_1^2p_2-3p_1^4].$$

Example 3. If α, β, γ be the roots of the cubic

$$x^3-px^2+qx-r=0,$$

find the value of $\Sigma\dfrac{1}{\alpha\beta}$.

We have $\alpha+\beta+\gamma=p$ and $\alpha\beta\gamma=r$.

Dividing the first of these relations by the second, we get

$$\Sigma\frac{1}{\alpha\beta}=\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}=\frac{p}{r}.$$

Example 4. Find the numerical value of $\Pi(\alpha^2+3)$, where $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$f(x)\equiv x^4+4x^3-6x^2-13x+11=0.$$

We have $x^4+4x^3-6x^2-13x+11$

$$\equiv(x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

Dividing $f(x)$ by (x^2+3) we get

$$f(x)\equiv(x^2+3)(x^2+4x-9)-25x+38\,;$$
$$\equiv(x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

$$\begin{array}{r|l}
 & 1+4-6-13+11 \\
0 & \;\;\;+0-3 \\
-3 & \;\;\;\;\;\;\;+0-12 \\
 & \;\;\;\;\;\;\;\;\;\;\;+\,0+27 \\
\hline
 & 1+4-9-25+38
\end{array}$$

Putting $i\sqrt{3}$ and $-i\sqrt{3}$ in succession for x in this identity, we have

$$38-25i\sqrt{3}=(i\sqrt{3}-\alpha)(i\sqrt{3}-\beta)(i\sqrt{3}-\gamma)(i\sqrt{3}-\delta)\,; \qquad (i)$$

and $$38+25i\sqrt{3}=(-i\sqrt{3}-\alpha)(-i\sqrt{3}-\beta)(-i\sqrt{3}-\gamma)(-i\sqrt{3}-\delta) \qquad (ii)$$

Multiplying (i) and (ii), we get

$$(\alpha^2+3)(\beta^2+3)(\gamma^2+3)(\delta^2+3)=38^2+3.25^2=3319.$$

Example 5. If α, β, γ be the roots of the cubic $x^3+qx+r=0$, find the value of $\Sigma\frac{\alpha^2+\beta\gamma}{\beta+\gamma}$.

We have $\Sigma\alpha=\alpha+\beta+\gamma=0$, $\Sigma\alpha\beta=\alpha\beta+\alpha\gamma+\beta\gamma=q$ and $\alpha\beta\gamma=-r$.

Therefore $$\Sigma\frac{\alpha^2+\beta\gamma}{\beta+\gamma}=\Sigma\left\{\frac{\alpha^2-\frac{r}{\alpha}}{-\alpha}\right\}$$

$$=\Sigma\left\{\frac{r}{\alpha^2}-\alpha\right\}=r\Sigma\frac{1}{\alpha^2}-\Sigma\alpha$$

$$=r\left\{\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2-2\left(\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\right)\right\}$$

$$=r\left\{\left(\frac{\Sigma\alpha\beta}{\alpha\beta\gamma}\right)^2-2\left(\frac{\Sigma\alpha}{\alpha\beta\gamma}\right)\right\}=\frac{q^2}{r}.$$

Example 6. Find the numerical value of $\Pi(2\alpha^2-5\alpha+3)$, where $\alpha, \beta, \gamma, \delta, \varepsilon$ are the roots of the equation :

$$f(x)\equiv 4x^5-2x^4-21x^2+2x-3=0.$$

Dividing $f(x)$ by $2x^2-5x+3$:

$$\begin{array}{r|l}
 & 4-\ 2+\ 0-21+\ 2-3 \\
5 & \ \ +10-\ 6 \\
-3 & \qquad\ \ +20-12 \\
 & \qquad\qquad\quad +35-21 \\
 & \qquad\qquad\qquad\quad +\ 5-3 \\
\hline
 & 4+\ 8+14+2 \ |-14-6 \\
\hline
2 & 2+\ 4+\ 7+1 \ | \\
\end{array}$$

we get $4(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\varepsilon)\equiv f(x)$

$$\equiv(2x^2-5x+3)(2x^3+4x^2+7x+1)-14x-6.$$

Putting for x in this identity, the values which make $2x^2-5x+3$ vanish, *i.e.* $x=1, 3/2$; we get

$$4(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon)=-20, \qquad (i)$$

and $4(\frac{3}{2}-\alpha)(\frac{3}{2}-\beta)(\frac{3}{2}-\gamma)(\frac{3}{2}-\delta)(\frac{3}{2}-\varepsilon)=-27.$ $\qquad (ii)$

Multiplying (*i*) and (*ii*), we get

$$16\ \Pi\ (\alpha^2-\tfrac{5}{2}\alpha+\tfrac{3}{2})=\tfrac{1}{2}\ \Pi\ (2\alpha^2-5\alpha+23)=540.$$

Therefore $\Pi\ (2\alpha^2-5\alpha+3)=1080.$

EXERCISES XII

1. If a, b, c be the roots of the cubic $x^3-px^2+qx-r=0$, find the value of (*i*) $\Sigma\frac{1}{a^2}$, (*ii*) $\Sigma\frac{1}{b^2c^2}$, (*iii*) Σa^2b^2, (*iv*) Σa^2b, (*v*) $\Sigma\left(\frac{b}{c}+\frac{c}{b}\right)$ and (*vi*) $(b+c)(c+a)(a+b)$.

Ans. (*i*) $\frac{q^2-2pr}{r^2}$, (*ii*) $\frac{p^2-2q}{r^2}$, (*iii*) q^2-2pr, (*iv*) $pq-3r$, (*v*) $\frac{pq-3r}{r}$, (*vi*) $pq-r$.

2. If a, b, c be the roots of $x^3+qx+r=0$,

find the value of (*i*) $\Sigma(b-c)^2$, (*ii*) $\Sigma\frac{1}{b+c}$,

$$(iii)\quad \Sigma a^3 \text{ and } (iv)\quad \Sigma\frac{b^2+c^2}{b+c}.$$

Ans. (*i*) $-6q$, (*ii*) $\frac{q}{r}$, (*iii*) $-3r$. (*iv*) $-\frac{2q^2}{r}$.

3. If a, b, c, d be the roots of the biquadratic

$$x^4-px^3+qx^2-rx+s=0,$$

find the value of (*i*) Σa^2bc, (*ii*) Σa^4, and (*iii*) Σa^3b.

Ans. (*i*) $pr-4s$, (*ii*) $p^4-4p^2q+2q^2+4pr-4s$,
(*iii*) $p^2q-2q^2-pr+4s$.

4. If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of the equation :

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\ldots\ldots+p_{n-1}x+p_n=0,$$

prove that $\prod_{r=0}^{n}(\alpha_r^2+1)=(1-p_2+p_4-p_6+\ldots)^2+(p_1-p_3+p_5-p_7+\ldots\ldots)^2$.

5. Find the numerical value of $\Pi\,(\alpha^2+3\alpha+2)$, α being a root of the equation : $x^4-4x^3+3x^2-17x+6=0$.

Ans. 3100.

§19. **Theorem.** *If* $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ *be the roots of the equation* $f_n(x)=0$, *then*

$$f'_n(x)=\frac{f_n(x)}{x-\alpha_1}+\frac{f_n(x)}{x-\alpha_2}+\frac{f_n(x)}{x-\alpha_3}+\ldots\ldots+\frac{f_n(x)}{x-\alpha_n}.$$

We have

$$f_n(x)\equiv p_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n).$$

Taking logarithmic differentiation, we get

$$\frac{f'_n(x)}{f_n(x)}=\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\frac{1}{x-\alpha_3}+\ldots\ldots+\frac{1}{x-\alpha_n},$$

Hence

$$f'_n(x)=\sum_{r=1}^{n}\frac{f_n(x)}{x-\alpha_r}.$$

§20. *To find the sum of the p^{th} powers of the roots of an equation, p being a positive or negative integer.*

Let $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots, \alpha_n$ be the roots of the equation

$$f_n(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots p_{n-1}x + p_n = 0.$$

We are required to find the value of $P_p \equiv \Sigma\alpha_r^{\ p}$ in terms of the co-efficients $p_1, p_2, p_3 \ldots\ldots p_n$. We have

$$f'_n(x) \equiv nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \ldots + p_{n-1}.$$

Moreover, by actual division, we obtain

$$\frac{f_n(x)}{x-\alpha_r} \equiv x^{n-1} + (\alpha_r + p_1)x^{n-2} + (\alpha_r^{\ 2} + p_1\alpha_r + p_2)x^{n-3} + \ldots\ldots$$

$$\ldots + (\alpha_r^{\ n-1} + p_1\alpha_r^{\ n-2} + p_2\alpha_r^{\ n-3} + \ldots\ldots + p_{n-1});$$

so that, from the Theorem of the preceding article,

$$nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \ldots\ldots + p_{n-1}$$

$$\equiv f'_n(x) \equiv \Sigma\frac{f_n(x)}{x-\alpha_r} \equiv nx^{n-1} + (np_1 + \Sigma\alpha_r)x^{n-2}$$

$$+ (np_2 + p_1\Sigma\alpha_r + \Sigma\alpha_r^{\ 2})x^{n-3} + \ldots + (np_{n-1} + p_{n-2}\Sigma\alpha_r$$
$$+ p_{n-3}\Sigma\alpha_r^{\ 2} + \ldots\ldots + \Sigma\alpha_r^{\ n-1}).$$

Equating the co-efficients on the two sides of this identity, we get $(n-1)p_1 = np_1 + \Sigma\alpha_r$ *i.e.*, $P_1 \equiv \Sigma\alpha_r = -p_1$;

$$(n-2)p_2 = np_2 + p_1\Sigma\alpha_r + \Sigma\alpha_r^{\ 2},$$

therefore $P_2 \equiv \Sigma\alpha_r^{\ 2} = p_1^{\ 2} - 2p_2$;

....

and in general,

$$(n-q)p_q = np_q + p_{q-1}\Sigma\alpha_r + p_{q-2}\Sigma\alpha_r^{\ 2} + \ldots + p_1\Sigma\alpha_r^{\ q-1} + \Sigma\alpha_r^{\ q},$$
$$q = 1, 2, 3, \ldots\ldots, (n-1);$$

or $P_q + p_1P_{q-1} + p_2P_{q-2} + \ldots\ldots + p_{q-1}P_1 + qp_q = 0,$

$$q = 1, 2, 3, \ldots\ldots\ldots (n-1).$$

From these $(n-1)$ relations, we can calculate in sucession the values of $P_1, P_2, P_3 \ldots\ldots P_{n-1}$ in terms of the co-efficients $p_1, p_2, p_3, \ldots\ldots\ldots p_{n-1}$.

To find P_p for other values of p, we substitute $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\alpha_n$ in succession for x in the equation

$$x^{m-n}f_n(x) \equiv x^m + p_1x^{m-1} + p_2x^{m-2} + \ldots + p_nx^{m-n} = 0.$$

Adding, we get the relation

$$P_m + p_1P_{m-1} + p_2P_{m-2} + \ldots\ldots + p_nP_{m-n} = 0,$$

m being any positive or negative integer and $P_0 = \Sigma\alpha_r^0 = n$.

Putting $n, n+1, n+2\ldots\ldots$ for m, we obtain in succession the values of $P_n, P_{n+1}, P_{n+2}\ldots\ldots$ from the above relation.

Again, putting $n-1, n-2, n-3\ldots\ldots\ldots$ for m in the same relation, we can get the values of $P_{-1}, P_{-2}, P_{-3}\ldots\ldots$

Thus. we can calculate P_p for all positive and negative integral values of p.

Example. Find the values of P_{-2}, P_2 and P_7 for the equation $f(x) \equiv x^6 - 3x^5 + 2x^2 - 3x - 2 = 0$.

Here $6x^5 - 15x^4 + 4x - 3 \equiv f'(x)$

$$\equiv \Sigma \frac{f(x)}{x-\alpha}, \ \alpha \text{ being a root of } f(x) = 0,$$

$$\equiv 6x^5 + (P_1 - 18)x^4 + (P_2 - 3P_1)x^3 + (P_3 - 3P_2)x^2 + (P_4 - 3P_3 + 12)x + (P_5 - 3P_4 + 2P_1 - 18).$$

Equating co-efficients, we get

$P_1 - 18 = -15$, $P_2 - 3P_1 = 0$, $P_3 - 3P_2 = 0$, $P_4 - 3P_3 + 12 = 4$ and $P_5 - 3P_4 + 2P_1 - 18 = -3$.

These relations give

$P_1 = 3$, $P_2 = 9$, $P_3 = 27$, $P_4 = 73$ and $P_5 = 228$.

Substituting $\alpha_1, \alpha_2, \alpha_3\ldots\ldots\alpha_6$ for x in succession, in the equation $x^{m-6}f(x) \equiv x^m - 3x^{m-1} + 2x^{m-4} - 2x^{m-6} = 0$,

we obtain on addition, the relation

$$P_m - 3P_{m-1} + 2P_{m-4} - 3P_{m-5} - 2P_{m-6} = 0.$$

Putting $m = 6$, $P_6 = 3P_5 - 2P_2 + 3P_1 + 2P_0$

$$= 684 - 18 + 9 + 12 = 687.$$

Again, putting $m=7$, we have

$$P_7=3P_6-2P_3+3P_2+2P_1=2040.$$

Putting $m=5$ and 4 in succession, in the same relation, we get $P_5-3P_4+2P_1-3P_0-2P_{-1}=0$, giving $P_{-1}=-\frac{3}{2}$; and $P_4-3P_3+2P_0-3P_{-1}-2P_{-2}=0$, giving $P_{-2}=\frac{17}{4}$.

In numerical cases such as the one above, the calculation is most easily made by using the results of § 21.

EXERCISES—XIII

1. Calculate the values of P_{-4}, P_{-3}, P_3 and P_4 for the following equations :—

(*i*) $x^3+5x^2-6x+3=0$; (*ii*) $7x^2-3x-4=0$;

(*iii*) $x^5-6x^3-3x+4=0$; (*iv*) $x^6-7x+1=0$;

(*v*) $x^4+4x^3-6x+2=0$; (*vi*) $x^8-1=0$.

Ans. (*i*) $-70/9$, -3, -224, 1357 ;
(*ii*) $\frac{2657}{256}$, $-\frac{279}{64}$, $\frac{279}{343}$, $\frac{2657}{2401}$; (*iii*) $\frac{1233}{256}$, $\frac{315}{64}$, 0, 84 ;
(*iv*) 2401, 343, 0, 0 ; (*v*) 55, 21, -46, 152 ;
(*vi*) 0, 0, 0, 0.

2. Find the values of P_4, P_6 and P_{-4} for the following equations ;

(*i*) $x^5+px^4+qx^2+t=0$; (*ii*) $x^4+px^3+qx^2+rx+s=0$.

Ans. (*i*) p^4+4pq ; $p^6+6p^3q+3q^2+6pt$; $\dfrac{2q^2}{t^2}-\dfrac{4p}{t}$.

(*ii*) $p^4-4p^2q+4pr+2q^2-4s$; $p^6-6p^4q+6p^3r$
$+9p^2q^2-6p^2s-12pqr+3r^2-2q^3+6qs$; etc.

3. Find the values of P_2, P_{-2} and P_3 for the equation :
$$x^3+x^2-8x-12=0,$$
which has two equal roots. Verify your answer by solving the equation. **Ans.** 17, $\frac{11}{18}$, 11.

§ 21. From the theorem of section 19, we have

$$\frac{f'_n(x)}{f_n(x)}=\Sigma\frac{1}{x-\alpha_r}=\Sigma\left[\frac{1}{x}+\frac{\alpha_r}{x^2}+\frac{\alpha_r^2}{x^3}+\ldots+\frac{\alpha_r^p}{x^{p+1}}+R\right]$$

by actual division, $\quad =\frac{n}{x}+\frac{P_1}{x^2}+\frac{P_2}{x^3}+\ldots+\frac{P_p}{x^{p+1}}+\Sigma R$, R being the remainder after $(p+1)$ terms, involving powers of $\frac{1}{x}$ higher than $(p+1)$

We thus see that for all positive integral values of p, P_p is the co-efficient of x^{-p-1} in the quotient of $f'_n(x)$ by $f_n(x)$.

Again $\frac{f'_n(x)}{f_n(x)}\equiv-\Sigma\frac{1}{\alpha_r-x}=-\Sigma\left[\frac{1}{\alpha_r}+\frac{x}{\alpha_r^2}+\frac{x^2}{\alpha_r^3}+\ldots+\frac{x^{p-1}}{\alpha_r^p}\right.$ $\left.+R\right]$. where R involves powers of x higher than $(p-1)$,

so that for negative integral values of p, P_p is the co-efficient of x^{-p-1} in the quotient of $-f'_n(x)$ by $f_n(x)$. The process of division is most easily carried on by the method of synthetic division. The terms will be arranged in descending or ascending powers of x according as p is positive or negative.

Example. Find the values of P_{-2}, P_2 and P_7 for the equation $f(x)\equiv x^6-3x^5+2x^2-3x-2=0$.

Here $f'(x)=6x^5-15x^4+4x-3$.

Dividing $f'(x)$ by $f(x)$:

$$\begin{array}{r|l}
 & 6-15+0+\ 0+\ 4-\ \ 3 \\
3 & \quad +18+0+\ 0-12+\ 18+\ 12 \\
0 & \qquad\qquad +9+\ 0+\ 0-\ \ 6+\ \ 9+\ \ \ 6 \\
0 & \qquad\qquad\qquad +27+\ 0+\ \ 0-\ 18+\ \ 27+\ldots \\
-2 & \qquad\qquad\qquad\qquad +81+\ \ 0+\ \ 0-\ \ 54+\ldots \\
3 & \qquad\qquad\qquad\qquad\qquad +219+\ \ 0+\ \ \ 0+\ldots \\
2 & \qquad\qquad\qquad\qquad\qquad\qquad +684+\ \ \ 0+\ldots \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad +2061+\ldots \\
\hline
1 & 6+\ 3+9+27+73+228+687+2040+\ldots
\end{array}$$

we get P_2=co-efficient of x^{-3} in $f'(x)/f(x)=9$;
and $\quad P_7$=co-efficient of x^{-8} in $f'(x)/f(x)=2040$.

To find P_{-2}, we expand $-f'(x)/f(x)$ in ascending powers of x.

Dividing $3-4x+15x^4-6x^5$ by $-2-3x+2x^2-3x^5+x^6$:

$$\begin{array}{r|l} & 3-\ 4+\dots\dots\\ 3 & \quad -\ \frac{9}{2}+\dots\dots \\ -2 & \\ : & \\ \hline -2 & 3-\frac{17}{2}+\dots\dots \\ \hline & -\frac{3}{2}+\frac{17}{4}+\dots\dots\end{array}$$

we have P_{-2}=co-efficient of x in $-f'(x)/f(x)=\frac{17}{4}$.

EXERCISES XIV

Calculate the values of P_{-5}, P_{-3}, P_0, P_3 and P_5 for each of the following equations :—

1. $x^4-3x^3+2x^2-3x+1=0$. **Ans.** 123; 18; 4; 18; 123.
2. $x^7-3x^3-2x-1=0$. **Ans.** -92; -17; 7; 0; 0.
3. $x^5-1=0$. **Ans.** 5; 0; 5; 0; 5.
4. $x^3-3x^2+3x-1=0$. **Ans.** 3; 3; 3; 3; 3.
5. $x^2-3x+2=0$. **Ans.** $\frac{33}{32}$; $\frac{9}{8}$; 2; 9; 33.

Verify your results in the case of the last two equations.

§ 22. Calculation of other Symmetric Functions.

Every rational integral symmetric function of the roots of an algebraic equation can be expressed rationally in terms of the co-efficients.

Every rational integral function in which the roots $\alpha_1, \alpha_2, \alpha_3 \dots\dots\dots, \alpha_n$ are symmetrically involved is of the form :

$$\Sigma\, \alpha_1{}^p \alpha_2{}^q \alpha_3{}^r \dots\dots\dots$$

Now $\Sigma\, \alpha_1{}^p\alpha_2{}^q = P_pP_q - P_{p+q}$, where $P_m=\Sigma\alpha_r{}^m$ and $p \neq q$.

For $P_pP_q=(\alpha_1{}^p+\alpha_2{}^p+\alpha_3{}^p+\dots\dots\dots+\alpha_n{}^p)\times$
$(\alpha_1{}^q+\alpha_2{}^q+\alpha_3{}^q+\dots\dots\dots+\alpha_n{}^q)$

$$=\Sigma\alpha_1{}^p\alpha_2{}^q+\Sigma\alpha_1{}^{p+q} \quad =\Sigma\alpha_1{}^p\alpha_2{}^q+P_{p+q}\ ;$$

so that $\Sigma\alpha_1{}^p\alpha_2{}^q=P_pP_q-P_{p+q}$ when $p\neq q$ ....(i)

Multiplying (i) by $P_r(\equiv\Sigma\alpha^r)$, we get

$$\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r+\Sigma\alpha_1{}^{p+r}\alpha_2{}^q+\Sigma\alpha_1{}^p\alpha_2{}^{q+r}=P_pP_qP_r-P_{p+q}P_r;$$

provided p, q, r are all different.

Since $\Sigma\alpha_1^{p+r}\alpha_2^{q}=P_{p+r}P_q-P_{p+q+r}$, if $q\neq p+r$;
and $\Sigma\alpha_1^{p}\alpha_2^{q+r}=P_pP_{q+r}-P_{p+q+r}$, if $p\neq q+r$;
we get $\Sigma\alpha_1^{p}\alpha_2^{q}\alpha_3^{r}=P_pP_qP_r-P_pP_{q+r}-P_{p+r}P_q$
$-P_{q+p}P_r+2P_{p+q+r}$.

Proceeding in this manner, we can always calculate the symmetric functions in terms of the P's, which are themselves expressible in terms of the co-efficients. Hence the proposition. The results have to be modified when any of the exponents become equal. Thus if $p=q$ we have

$$(P_p)^2=(\alpha_1^p+\alpha_2^p+\alpha_3^p+\ldots\ldots+\alpha_n^p)=\Sigma\alpha_1^{2p}+2\Sigma\alpha_1^p\alpha_2^p.$$

Hence $$\Sigma\alpha_1^p\alpha_2^p=[(P_p)^2-P_{2p}]/2. \qquad (ii)$$

Similarly if $p=q=r$ then

$$\Sigma\alpha_1^p\alpha_2^p\alpha_3^p=\frac{1}{3\,!}\,(P_p^3-3P_pP_{2p}+2P_{3p}) \qquad (iii)$$

since the six terms obtained by interchanging the roots in $\alpha_1^p\alpha_2^q\alpha_3^r$ become all equal.

The fundamental results (*i*) and (*ii*) can together be expressed in the form :

$$\Sigma\alpha_1^p\alpha_2^q=\frac{3\pm1}{4}[P_pP_q-P_{p+q}]$$

according as $p\neq q$ or $p=q$.

A few examples will illustrate the points involved.

Example 1. Find the value of $\Sigma\alpha^2\beta\gamma$,

α, β, γ, δ being the roots of the biquadratic

$$x^4+px^3+qx^2+rx+s=0.$$

We have $\Sigma\alpha^2\beta=P_2P_1-P_3$.

Multiplying both sides by P_1, we get

$$2(\Sigma\alpha^2\beta\gamma+\Sigma\alpha^2\beta^2)+\Sigma\alpha^3\beta=P_2P_1^2-P_3P_1.$$

But $\Sigma\alpha^3\beta=P_3P_1-P_4$ and $2\Sigma\alpha^2\beta^2=P_2^2-P_4$.

Therefore $2\Sigma\alpha^2\beta\gamma=P_2P_1^2-2P_3P_1+2P_4-P_2^2$.

Dividing $4x^3+3px^2+2qx+r$ by $x^4+px^3+qx^2+rx+s$;

$$\begin{array}{r|lllll}
 & 4, & +3p, & +2q, & +r & \\
-p & & -4p, & -4q, & -4r, & -4s \\
-q & & & +p^2, & +pq, & +pr \quad \ldots\ldots\ldots\ldots \\
-r & & & & -p^2+2pq, & -p^2q+2q^2\ldots\ldots\ldots\ldots \\
-s & & & & & +p^4-3p^2q+3p^2q+3pr\ \ldots \\
\hline
1 & 4, & -p, & p^2-2q, & -p^3+3pq-3r, & p^4-4p^2q+2q^2+4pr-4s\ldots \\
\hline
\end{array}$$

we have $P_1=-p$, $P_2=p^2-2q$, $P_3=-p^3+3pq-3r$ and
$P_4=p^4-4p^2q+2q^2+4pr-4s$.

Substitution gives $\Sigma\alpha^2\beta\gamma=pr-4s$.

Example 2. Find the numerical value of $\Sigma\alpha^3\beta^2$ for the equation $x^5-3x^3+2=0$.

Here $\Sigma\alpha^3\beta^2=P_3P_2-P_5$.

Dividing $5x^4-9x^2$ by x^5-3x^3+2.

$$\begin{array}{r|l}
 & 5+0-9+0+0 \\
0 & \quad +0+15+0+0-10 \\
3 & \qquad +0+0+0+0+\ldots\ldots \\
0 & \qquad\quad +0+18+0+\ldots\ldots \\
0 & \qquad\qquad +0+0+\ldots\ldots \\
-2 & \qquad\qquad\quad +0+\ldots\ldots \\
\hline
1 & 5+0+6+0+18-10+\ldots\ldots \\
\hline
\end{array}$$

we have $P_1=0$, $P_2=6$, $P_3=0$, $P_4=18$, $P_5=-10$.

Therefore $\Sigma\alpha^3\beta^2=10$.

Example 3. Find the numerical value of $\Sigma\alpha^2\beta\gamma$ for the cubic $x^3-3x^2+3x-1=0$.

As in example 1, $2\Sigma\alpha^2\beta\gamma=P_2P_1^2-2P_3P_1-P_2^2+2P_4$.

Dividing $3x^2-6x+3$ by x^3-3x^2+3x-1.

$$\begin{array}{r|l}
 & 3-6+3 \\
3 & \quad +9-9+3 \\
-3 & \qquad +9-9+3 \\
+1 & \qquad\quad +9-9+\ldots\ldots \\
 & \qquad\qquad +9-\ldots\ldots \\
\hline
1 & 3+3+3+3+3+ \\
\hline
\end{array}$$

we obtain $P_1=P_2=P_3=P_4=3$.

Hence $\Sigma\alpha^2\beta\gamma=3$.

Or $\Sigma\alpha^2\beta\gamma=\alpha\beta\gamma(\alpha+\beta+\gamma)=1.3=3$.

EXERCISES XV

Find the numerical values of

(i) $\Sigma\alpha^2\beta^2$, (ii) $\Sigma\alpha^3\beta$, (iii) $\Sigma\alpha^2\beta\gamma$

for the following equations :—

1. $x^6+3x^5+3x^3-7=0$. **Ans.** 18 ; 9 ; -9.
2. $x^7-1=0$. **Ans.** 0 ; 0 ; 0.
3. $x^3-5x+1=0$. **Ans.** 25 ; -50 ; 0.
4. $x^4-5x^3+2x^2-x-4=0$. **Ans.** -14 ; 21 ; 21.

§ 23. Order and Weight of a Symmetric Function.

The weight of a symmetric function of the roots is the degree in all the roots, of any term of the function. The order of a symmetric function of the roots is the highest degree in which each root enters the function. Thus, for example, the weight of $\Sigma\alpha^3\beta^2\gamma^4$ is $3+2+4$ *i.e.*, 9 and its order is 4. From the relations between the roots and co-efficients of an equation, it can be inferred that the degree in terms of the co-efficients $p_1, p_2, p_3 \ldots p_n$ of the value of any symmetric function is equal to the order of the symmetric function. Thus, for example, if α, β, γ be the roots of the cubic $x^3+p_1x^2+p_2x+p_3=0$, then

$$\Sigma\alpha^2\beta=3p_3-p_1p_2.$$

The expression on the right hand side is of degree two in the co-efficients p_1 and p_2. This is the same as the order of the symmetric function $\Sigma\alpha^2\beta$.

§ 24. Homogeneous Products. The sum of all the symmetric functions of weight r, which can be formed from any n letters is called the "sum of the homogeneous products of r dimensions" of the n letters and is denoted by $\underset{r}{\Pi}$.

It follows from the definition that $\underset{r}{\Pi}$ is the co-efficient of

$$x^r \text{ in } (1+\alpha_1x+\alpha_1^2x^2+\ldots\ldots)(1+\alpha_2x+\alpha_2^2x^2+\ldots\ldots)\ldots\ldots(1+\alpha_nx+\alpha_n^2x^2+\ldots\ldots).$$

EXERCISES XVI

1. Express the sums of the homogeneous products of the roots in terms of the co-efficients of an equation and *vice versa.*

[Hint. $1+\underset{1}{\Pi}x+\underset{2}{\Pi}x^2+\ldots\equiv\dfrac{1}{1+p_1x+p_2x^2+\ldots p_nx^n}$].

2. Express $\underset{r}{\Pi}$ in terms of the sums of the powers of the roots.

[Hint. $1+\underset{1}{\Pi}x+\underset{2}{\Pi}x^2+\ldots\equiv\dfrac{1}{(1-\alpha_1x)(1-\alpha_2x)\ldots(1-\alpha_nx)}$.

Take logarithmic differentiation].

3. Show that in the equation

$$f(x)\equiv x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0,$$

the value of P_r expressed in terms of the co-efficients $p_1, p_2, p_3, \ldots\ldots p_n$ is the co-efficient of y^r in the expansion, by ascending powers of y, of $-r\log[y^nf(\frac{1}{y})]$.

Solution. Putting $\frac{1}{y}$ for x in the identity

$$f(x)\equiv(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n),$$

we get $1+p_1y+p_2y^2+\ldots+p_ny^n\equiv(1-\alpha_1y)(1-\alpha_2y)(1-\alpha_3y)\ldots\ldots(1-\alpha_ny)$.

Taking logarithms of both sides, we have

$$\log[y^n f(\frac{1}{y})]=\log(1-\alpha_1y)+\log(1-\alpha_2y)+(1-\alpha_3y)+\ldots\ldots+\log(1-\alpha_ny)\ldots\ldots(i)$$

Since $\log\ (1-\alpha y)=-(\alpha y+\frac{\alpha^2y^2}{2}+\frac{\alpha^3y^3}{3}+\ldots+\frac{\alpha^ry^r}{r}+\ldots\ldots)$,

the co-efficient of y^r on the right hand side of (i) is $-\frac{P_r}{r}$.

Hence the result.

4. Express the co-efficients of the equation

$$f_n(x)\equiv x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0,$$

in terms of the sums of the powers of the roots.

Solution. We have

$$1+p_1y+p_2y^2+\ldots+p_ny^n\equiv y^nf(\frac{1}{y})\equiv e^{\log[y^nf\,(\frac{1}{y})]}$$

$$\equiv e^{-P_1y-\frac{P_2}{2}y^2-\frac{P_3}{3}y^3-\ldots-\frac{P_r}{r}y^r-\ldots}$$

$$\equiv 1-(P_1y+\frac{P_2}{2}y^2+\frac{P_2}{3}y^3+\ldots)$$

$$+\frac{1}{2!}(P_1y+\frac{P_2}{2}y^2+\ldots)^2-\ldots\ldots$$

Equating co-efficients of like powers of y, we get

$$p_1=-P_1,\ p_2=-\frac{P_2}{2}+\frac{P_1^2}{2\,!},\ \text{and so on.}$$

5. Prove that $\dfrac{1}{f_n(x)}=\Sigma\dfrac{1}{f_n'(\alpha_r)}\cdot\dfrac{1}{x-\alpha_r}$,

where $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ are the roots of the equation $f_n(x)=0$.

Show that $\dfrac{x^n}{f_n(x)}=\Sigma\dfrac{\alpha_r^{\,n-1}}{f_n'(\alpha_r)}\cdot\dfrac{1}{1-\alpha_ry}$, where $y=\dfrac{1}{x}$.

Hence deduce that

$$\underset{m}{\Pi}=\Sigma\frac{\alpha_r^{\,n+m-1}}{f_n'(\alpha_r)}.$$

6. Express P_5 in terms of the co-efficients $p_1, p_2, p_3, \ldots, p_n$ of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$,

Express $\underset{3}{\Pi}$ in the same way and show that

$$\frac{\partial P_5}{\partial p_2}=-5\underset{3}{\Pi}.$$

7. Calculate the value of $\underset{4}{\Pi}$ for the equation

$$x^5-3x^3-2x+4=0.$$ **Ans.** 83.

C.T.M.: Reciprocal ... replaced by ...
unchanged when x is replaced by $\frac{1}{x}$
Types:- i) Those in which co. of terms equi-
distant from beginning & the end be
same & of same sign e.g. ...
ii) Those in which these co. are equal
but of opposite sign e.g. ...

CHAPTER IV

Transformation of Equations

§ **25.** Given an equation, we can generally transform it into another, whose roots shall have certain assigned relations with the roots of the given equation. The transformed equation may be easier to solve or it may clearly exhibit certain other facts which may not be quite evident from the given equation. Transformation may thus facilitate the discussion of a given equation. In the present Chapter, we shall give methods for affecting elementary transformations.

§ **26.** *To transform an equation into another, whose roots shall be equal in magnitude but opposite in sign to those of the given equation.*

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of the equation :

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0.$$

Then $p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n$

$$\equiv p_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n).$$

Put $-x$ for x in this identity, then after simplification, we get $\quad p_0x^n-p_1x^{n-1}+p_2x^{n-2}\ldots\ldots+(-1)^np_n$

$$\equiv p_0(x+\alpha_1)(x+\alpha_2)(x+\alpha_3)\ldots\ldots(x+\alpha_n).$$

The right hand side of this identity vanishes when x has any of the values $-\alpha_1, -\alpha_2, \ldots\ldots\ldots -\alpha_n$.

Therefore, the required equation is

$$p_0x^n-p_1x^{n-1}+p_2x^{n-2}-\ldots\ldots+(-1)^np_n=0.$$

Example. Transform the equation :

$$x^7-7x^6-3x^4+4x^2-3x-2=0,$$

into another whose roots shall be equal in magnitude but opposite in sign to those of this equation.

Changing x into $-x$, we get the required equation

$$-x^7-7x^6-3x^4+4x^2+3x-2=0,$$

or
$$x^7+7x^6+3x^4-4x^2-3x+2=0,$$

EXERCISES XVII

Apply the transformation of § 26 to the following equations :—

1. $x^5+4x^4-3x^3-2x^2-x+5=0$.
   Ans. $x^5-4x^4-3x^3+2x^2-x-5=0$.

2. $x^6-x^4+x^2-1=0$. **Ans.** $x^6-x^4+x^2-1=0$.

3. $x^3+3x^2+4x-1=0$. **Ans.** $x^3-3x^2+4x+1=0$.

3. $x^5-x^3+x^2-1=0$. **Ans.** $x^5-x^3-x^2+1=0$.

Solve the following equations which have some pairs of roots equal in magnitude but opposite in sign.

5. $x^5-3x^4-5x^3+15x^2+4x-12=0$. **Ans.** $\pm 1, \pm 2, 3$.

6. $x^3-25+x(x-25)=0$. **Ans.** $-1, \pm 5$.

§ **27.** *To transform an equation into another whose roots shall be m times the roots of the given equation.*

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of the equation :

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0.$$

Then $\sum\limits_{r=0}^{n} p_rx^{n-r} \equiv p_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n)$.

Changing x into x/m in this identity and multiplying both sides by m^n, we get

$$\sum_{r=0}^{n} p_rm^rx^{n-r} \equiv p_0 \prod_{k=1}^{n} (x-\alpha_k m).$$

The right hand side of this identity vanishes when

$$x=m\alpha_k,\ k=1, 2, 3, \ldots\ldots\ldots, n.$$

Therefore, the required equation is

$$p_0x^n+p_1mx^{n-1}+p_2m^2x^{n-2}+\ldots\ldots p_nm^n=0.$$

The transformation is affected therefore, by multiplying the successive terms beginning with the second by $m, m^2, m^3, \ldots m^n$.

Example 1. Form an equation whose roots shall be three times the roots of the equation

$$3x^5-2x^4+x^3-x^2+2x-1=0.$$

The required equation is

$$3x^5-6x^4+9x^3-27x^2+162x-243=0\ ;$$

i.e. Nota Bene

$$x^5-2x^4+3x^3-9x^2+54x-81=0.$$

<u>*N.B.*</u> This transformation is helpful in getting rid of the fractional co-efficients in an equation, as will be seen from the following example.

Example 2. Transform the equation

$$5x^3-\tfrac{3}{2}x^2-\tfrac{3}{4}x+1=0.$$

into another with integral co-efficients and unity for the co-efficient of the first term.

The given equation can be put into the form :

$$x^3-\tfrac{3}{10}x^2-\tfrac{3}{20}x+\tfrac{1}{5}=0$$

Multiplying the roots of this equation by m, we get

$$x^3-\tfrac{3}{10}mx^2-\tfrac{3}{20}m^2x+\tfrac{1}{5}m^3=0.$$

The least value of m for which the fractions will disappear is 10.

Substituting 10 for m, we get the required equation

$$x^3-3x^2-15x+200=0.$$

EXERCISES XVIII

1. Form an equation whose roots shall be (*i*) five times, (*ii*) -6 times the roots of the equation : $x^3-4x^2+\frac{1}{2}x-\frac{1}{9}=0$.

Ans. (*i*) $x^3-20x^2+\frac{25}{2}x-\frac{125}{9}=0$.

(*ii*) $x^3+24x^2+18x+24=0$.

2. Transform the following equations into others with integral co-efficients and unity for the co-efficient of the first term :—

(*i*) $x^3+\frac{1}{2}x^2+\frac{1}{3}x+\frac{1}{4}=0$. **Ans.** $x^3+3x^2+12x+54=0$.

(*ii*) $3x^4-5x^3+x^2-x+1=0$; **Ans.** $x^4-5x^3+3x^2-9x+27=0$.

(*iii*) $\frac{2}{3}x^4+\frac{1}{4}x^3-x+\frac{1}{6}=0$. **Ans.** $x^4+3x^3-768x+1024=0$.

3. Show that the transformation of § 26 is only a particular case of that of § 27. Hence form an equation whose roots

shall be equal in magnitude but opposite in sign to those of the equation $x^6-5x^3-4x^2+4x+7=0$.

§ **28.** *To transform an equation into another whose roots shall be reciprocals of the roots of the given equation.*

We have (as in § 27).

$$\sum_{r=0}^{n} p_r x^{n-r} \equiv p_o \prod_{k=1}^{n} (x-\alpha_k).$$

In this identity, put $\frac{1}{x}$ for x and multiply both sides by x^n, then

$$\sum_{r=0}^{n} p_r x^r \equiv p_o \prod_{k=1}^{n} (1-\alpha_k x).$$

The right hand side of this identity vanishes when

$$x=\frac{1}{\alpha_k}, k=1, 2, 3, \ldots n.$$

Therefore, the required equation is

$$\sum_{r=0}^{n} p_r x^r = 0 \quad i.e.\ p_n x^n + p_{n-1} x^{n-1} + p_{n-2} x^{n-2} + \ldots\ldots$$

$$\ldots\ldots + p_1 x + p_o = 0.$$

Example. Form an equation whose roots shall be the reciprocals of the roots of the equation :

$$x^5-4x^3+6x^2-3x+2=0.$$

Putting $\frac{1}{x}$ for x and multiplying by x^5, we have the required equation, *viz.* $2x^5-3x^4+6x^3-4x^2+1=0$.

EXERCISES XIX

1. Apply the transformation of § 28 to the following equations (*i*) $x^4-3x^3+x^2-x+5=0$;

(*ii*) $4x^5-3x^4+2x^3-2x^2+3x-4=0$;

(*iii*) $x^6-4x^5+3x^4+4x^3+3x^2-4x+1=0$.

2. The roots of the equation

$$81x^3-18x^2-36x+8=0.$$

are in H. P. Transform it into another with integral co-efficients and unity for the co-efficient of the first term, so that the roots of the transformed equation may be in A. P.

Ans. $x^3-9x^2-9x+81=0$

3. Solve the equation of exercise 2, be solving the transformed equation. **Ans.** $-\frac{2}{3}, \frac{2}{3}, \frac{2}{9}$.

§ 29. **Reciprocal Equations**. The equations that remain unaltered when the variable x is changed into its reciprocal $\dfrac{1}{x}$, are termed "Reciprocal Equations." Such are, for example, the equations :

$$\text{(i)}\quad x^5-5x^4+6x^3-6x^2+5x-1=0$$

$$\text{(ii)}\quad x^6+6x^5-3x^4+3x^2-6x-1=0,$$

$$\text{and (iii)}\quad x^4-4x^3+8x^2-4x+1=0$$

§ 30. *To find the condition that the general equation of the n^{th} degree may be a reciprocal equation.*

If the equation : $p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0$, remains unaltered when x is changed into $\dfrac{1}{x}$, it must be the same as the equation :

$$p_nx^n+p_{n-1}x^{n-1}+p_{n-2}x^{n-2}+\ldots+p_1x+p_0=0$$

Comparing the co-efficients we, have

$$\frac{p_0}{p_n}=\frac{p_1}{p_{n-1}}=\frac{p_2}{p_{n-2}}=\ldots=\frac{p_r}{p_{n-r}}=\ldots=\frac{p_{n-r}}{p_r}=\ldots=\frac{p_{n-2}}{p_2}=\frac{p_{n-1}}{p_1}=\frac{p_n}{p_0}$$

Hence $p_n{}^2=p_0{}^2$, so that $p_n=p_0$ or $-p_0$.

If $p_n=p_0$. then $p_{n-1}=p_1, p_{n-2}=p_2, \ldots, p_{n-r}=p_r, \ldots\ldots\ldots\ldots$ *i. e.* the co-efficients of the terms equidistant from the beginning and the end are equal in magnitude and of the same sign.

If $p_n=-p_0$, then $p_{n-1}=-p_1, p_{n-2}=-p_2, \ldots p_{n-r}=-p_r\ldots\ldots$ so that the co-efficients of the terms equidistant from the beginning and the end are equal in magnitude but opposite in sign.

[illegible] In case = n is of odd degree then it will have one of its roots +1 or −1 and if it is even degree then it will have (x^2-1) as one of its roots/factors

In the latter case, if the equation be of an even degree say $2m$, the number of terms will be $2m+1$, the middle term will be the $(m+1)^{th}$ and its coefficient will be p_m. Then we get $p_m=-p_m$ or $2p_m=0$ or $p_m=0$.

The three equations given in § 29. illustrate the conclusions of this article. In equations (i) and (ii) the cofficients of terms equidistant from the beginning and the end are equal in magnitude but opposite in sign. Equation (ii) is of even degree and is therefore, wanting in the middle term. In equation (iii), the coefficients of terms equidistant from the beginning and the end are equal in magnitude and of the same sign.

§ 31. Standard form of Reciprocal Equations.

If the equation be of an odd degree, then one of the roots say α, must be its own reciprocal so that $\alpha=\frac{1}{\alpha}$ or $\alpha^2=1$ $\therefore \alpha=+1$ or -1 *i.e.* $(x-1)$ or $(x+1)$ is a factor of $f_n(x)$.

If the coefficients of terms equidistant from the beginning and the end have like signs, the equation is of the form

$$p_0x^{2m+1}+p_1x^{2m}+\ldots+p_mx^{m+1}+p_mx^m+\ldots+p_1x+p_0=0$$

which can be writen as

$$p_0(x^{2m+1}+1)+p_1x(x^{2m-1}+1)+\ldots+p_mx^m(x+1)=0.$$

By substitution we see that -1 is a root and hence $(x+1)$ is a factor of $f_n(x)$.

If the co-efficients of terms equidistant from the beginning and the end have unlike signs, the equation is of the form

$$p_0x^{2m+1}+p_1x^{2m}+\ldots+p_mx^{m+1}-p_mx^m\ldots-p_1x-p_0=0$$

$$\text{or } p_0(x^{2m+1}-1)+p_1x(x^{2m-1}-1)+\ldots+p_mx^m(x-1)=0$$

which is satisfied by $x=1$ and hence $(x-1)$ is a factor of $f_n(x)$.

In either case, the degree of the equation can be depressed by unity, by dividing $f_n(x)$ by $(x+1)$ or $(x-1)$ as the case may be. The depressed equation is a reciprocal equation of an even

degree, the co-efficients of terms equidistant from the beginning and the end having like signs.

If the given reciprocal equation $f_n(x)=0$ be of an even degree and the co-efficients of terms equidistant from the beginning and the end be equal in magnitude but opposite in sign, the equation can be written as

$$p_0(x^{2m}-1)+p_1x(x^{2m-2}-1)+\ldots\ldots+p_{m-1}x^{m-1}(x^2-1)=0,$$

all the terms on the left hand side occurring in pairs, which is satisfied by $x=+1$ and $x=-1$ and hence (x^2-1) is a factor of $f_n(x)$. The degree of the equation can, in this case, be depressed by two. The depressed equation is again a reciprocal equation of an even degree with like signs of the co-efficients of the terms equidistant from the beginning and the end. This is the 'Standard Form' to which all reciprocal equations can be reduced. The process may be repeated if necessary.

Example. Reduce the following reciprocal equations to the standard form :—(*i*) $6x^6+5x^5-44x^4+44x^2-5x-6=0$;
(*ii*) $7x^5-5x^4-2x^3+2x^2+5x-7=0$;
(*iii*) $7x^5-5x^4-2x^3-2x^2-5x+7=0$.

Equation (*i*) is a reciprocal equation of an even degree with unlike signs of the co-efficients equidistant from the beginning and the end.

Dividing the expression on the left by (x^2-1),

$$\begin{array}{r|l}
 & 6+5-44+\ 0+44-5-6 \\
-1 & \quad -6+\ 1+43-43-1+6 \\ \hline
 & 6-1-43+43+\ 1-6 \;+0 \\
1 & \quad +6+\ 5-38+\ 5+6 \\ \hline
 & 6+5-38+\ 5+\ 6 \;+0
\end{array}$$

we get $$6x^4+5x^3-38x^2+5x+6=0,$$

as the standard form of the equation.

Equation (*ii*) is of an odd degree, and the co-efficients equidistant from the beginning and the end have unlike signs.

The expression on the left has, therefore, to be divided by $(x-1)$ to obtain the standard form,

$$\begin{array}{r|l} & 7-5-2+2+5\ -7 \\ 1 & \quad +7+2+0+2\ +7 \\ \hline & 7+2+0+2+7 \,|+0 \end{array}$$

Thus the required equation is

$$7x^4+2x^3+2x+7=0.$$

Equation (*iii*) is reduced to the standard form. when the expression on the left is divided by $(x+1)$

$$\begin{array}{r|l} & 7-\ 5-\ 2-\ 2-\ 5\ +7 \\ -1 & \quad -\ 7+12-10+12\ -7 \\ \hline & 7-12+10-12+\ 7 \,|+0 \end{array}$$

This gives the equation

$$7x^4-12x^3+10x^2-12x+7=0.$$

EXERCISES XX

Reduce the following reciprocal equations to their standard forms :—

1. $x^5-x^4-x^3-x^2-x+1=0.$
   Ans. $x^4-2x^3+x^2-2x+1=0.$

2. $x^4+x^3+x^2+x+1=0.$ **Ans.** $x^4+x^3+x^2+x+1=0.$

3. $x^5+x^4+x^3-x^2-x-1=0.$
   Ans. $x^4+2x^3+3x^2+2x+1=0.$

4. $x^6+x^5+x^4-x^2-x-1=0.$
   Ans. $x^4+x^3+2x^2+x+1=0.$

§ 32. *A Reciprocal Equation of the Standard Form can be reduced to an equation of half its dimensions.*

Let $p_0x^{2m}+p_1x^{2m-1}+p_2x^{2m-2}+\ldots\ldots+p_mx^m+\ldots\ldots$
$$+p_2x^2+p_1x+p_0=0$$
be the given equation.

Dividing by x^m and re-arranging the terms, we have

$$p_0(x^m+\frac{1}{x^m})+p_1(x^{m-1}+\frac{1}{x^{m-1}})+p_2(x^{m-2}+\frac{1}{x^{m-2}})+\dots$$
$$+p_{m-1}(x+\frac{1}{x})+p_m=0.$$

In this equation, let $x+\frac{1}{x}$ be equal to z.

Then, denoting $x^r+\frac{1}{x^r}$ by u_r, we have the identical relation :

$u_{r+1}\equiv zu_r-u_{r-1}$ for $x^{r+1}+\frac{1}{x^{r+1}}\equiv(x+\frac{1}{x})(x^r+\frac{1}{x^r})-(x^{r-1}+\frac{1}{x^{r-1}})$

Since $u_0=x^0+\frac{1}{x^0}=2$, and $u_1=x+\frac{1}{x}=z$,

this relation enables us to calculate the values of u_2, u_3, u_4...... in succession. Thus

$$u_2=zu_1-u_0=z^2-2\ ;$$
$$u_3=zu_2-u_1=z^3-3z\ ;$$
$$u_4=zu_3-u_2=z^4-4z^2+2\ ;$$

and so on*. In general u_r is of the rth degree in z.

The equation $p_0u_m+p_1u_{m-1}+p_2u_{m-2}+\dots+p_{m-1}u_1+p_m=0$, to which the standard reciprocal equation has been reduced is, thus, of the mth degree in z.

To every root of this reduced equation in z, correspond two roots of the reciprocal equation. Thus, if k be a root of the reduced equation, the quadratic

$$x+\frac{1}{x}=k \text{ i.e., } x^2-kx+1=0,$$

* Since $x+\frac{1}{x}=z$, $\left(x-\frac{1}{x}\right)^2=z^2-4$, or $x-\frac{1}{x}=\sqrt{z^2-4}$

Therefore $u_r=x^r+\frac{1}{x^r}=\left[\frac{z+\sqrt{z^2-4}}{2}\right]^r+\left[\frac{z-\sqrt{z^2-4}}{2}\right]^r$

gives the two corresponding roots $\dfrac{k\pm\sqrt{k^2-4}}{2}$ of the given reciprocal equation.

Example. Solve the equation

$$f(x)\equiv 4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0.$$

which has some equal roots.

Dividing $f(x)$ by x^3 and re-arranging the terms, we have

$$4\left(x^3+\frac{1}{x^3}\right)-24\left(x^2+\frac{1}{x^2}\right)+57\left(x+\frac{1}{x}\right)-73=0.$$

Putting $x+\dfrac{1}{x}=z$, we get the cubic

$$\begin{aligned} F(z) &\equiv 4(z^3-3z)-24(z^2-2)+57z-73=0 \\ \text{or} \quad & 4z^3-24z^2+45z-25=0. \end{aligned}$$

If this has equal roots, $F(z)$ and $F'(z)$ have a common factor. This is found as follows :—

$$\begin{array}{r|r|r|l} & 3\,)\,12-48+45 & & \\ & 4-16+15 & 4-24+45-25 & 1 \\ 2 & 4-10 & 4-16+15 & \\ \hline & -6+15 & -8+30-25 & -2 \\ -3 & -6+15 & -8+32-30 & \\ \hline & \times & -1)\,-2+5 & \\ & & 2-5 & \end{array}$$

Thus the H.C.F. of $F(z)$ and $F'(z)$ is $2z-5$.

Dividing $F(z)$ by $(2z-5)^2$, we get $F(z)\equiv(2z-5)^2(z-1)$.

$$\begin{array}{r|rr} & 4-24 & +45-25 \\ 20 & +20 & -25 \\ -25 & & -20+25 \\ \hline 4 & 4-4 & +0+\ 0 \\ & 1-1 & \end{array}$$

Thus the roots of the equation

$$F(z)=0.$$

are $\frac{5}{2}.\ \frac{5}{2},\ 1$

Therefore, the roots of the given reciprocal equation are :

$$\frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2},\quad \frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2},\quad \frac{1\pm\sqrt{1-4}}{2};$$

i.e., $\quad 2,\ \frac{1}{2},\ 2,\ \frac{1}{2}\ \dfrac{1\pm i\sqrt{3}}{2}.$

EXERCISES XXI

Solve :

1. $2x^4+x^3-6x^2+x+2=0$ **Ans.** $1, 1, -\frac{1}{2}, -5$.
2. $x^5-5x^4+9x^3-9x^2+5x-1=0$.

Ans. $1, \dfrac{3\pm\sqrt{5}}{2}, \dfrac{1\pm i\sqrt{3}}{2}$

3. $x^6+2x^5+2x^4-2x^2-2x-1=0$.

Ans. $1, -1, \dfrac{-1\pm i\sqrt{3}}{2}, \dfrac{1\pm i\sqrt{3}}{2}$.

4. $ax^5+(b-ac)x^4-bcx^3-bx^2-(a-bc)x+ac=0$, c being a root.

Ans. $c, 1, -1, \dfrac{-b\pm\sqrt{b^2-4a^2}}{2a}$

§ **33.** *To form an equation whose roots shall be the squares of those of a given equation.*

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ be the roots of the equation

$$f(x)\equiv\sum_{r=0}^{n}p_r x^{n-r}=0.$$

Then
$$f(x)\equiv p_0\prod_{k=1}^{n}(x-\alpha_k), \qquad \ldots(i)$$

Changing x into $-x$ in (i), we get

$$f(-x)\equiv\sum_{r=0}^{n}p_r(-x)^{n-r}\equiv p_0\prod_{k=r}^{n}(-x-\alpha_k)$$

$$\equiv(-1)^n p_0\prod_{k=1}^{n}(x+\alpha_k). \qquad \ldots(ii)$$

Multiplying (i) and (ii), we have

$$\sum_{r=0}^{n}p_r x^{n-r}.\sum_{r=0}^{n}(-1)^r p_r x^{n-r}\equiv p_0^2\prod_{r=1}^{n}(x^2-\alpha_k^2).$$

i.e. $(p_0x^n+p_2x^{n-2}+p_4x^{n-4}+\ldots+p_{2t}x^{n-2t}+\ldots)^2-(p_1x^{n-1}+p_3x^{n-3}+p_5x^{n-5}+\ldots)^2$

$$\equiv p_0^2(x^2-\alpha_1^2)(x^2-\alpha_2^2)(x^2-\alpha_3^2)\ldots(x^2-\alpha_n^2). \qquad \ldots(iii)$$

The right hand side of (*iii*) is a function of x^2, so also must be the left hand side.

Replacing x^2 by x in (*iii*), we finally obtain

$$p_0^2x^n-(p_1^2-2p_0p_2)x^{n-1}+\ldots\ldots+(-1)^np_n^2$$
$$\equiv p_0^2(x-\alpha_1^2)(x-\alpha_2^2)(x-\alpha_3^2)\ldots\ldots(x-\alpha_n^2). \quad \ldots(iv)$$

The right hand side of (*iv*) vanishes when

$$x=\alpha^2_k,\ k=1, 2, 3\ldots\ldots n.$$

Therefore, the required equation is

$$p_0^2x^n-(p_1^2-2p_0p_2)x^{n-1}+\ldots\ldots+(-1)^np_n^2=0.$$

?

Example. Form an equation whose roots shall be the squares of the roots of the equation

$$f(x)\equiv x^5-3x^4+2x^3-3x^2-5=0.$$ consult Hem Raj's Book page 26 Example (ii e)

The required equation is obtained by changing x^2 into x in the equation :

$$f(x).f(-x)=0\ ;$$

or

$$(x^5+2x^3)^2-(-3x^4-3x^2-5)^2=0.$$

We thus have the equation

$$x(x^2+2x)^2-(-3x^2-3x-5)^2=0.$$

i.e.,

$$x^5-5x^4-14x^3-39x^2-30x-25=0.$$

EXERCISES XXII

1. Form equations whose roots shall be the squares of those of the following equations :—

(*a*) $x^5+x^3+x^2+2x+3=0.$
Ans. $x^5+2x^4+5x^3+3x^2-2x-9=0.$

(*b*) $x^3+3x^2+3x+1=0.$
Ans. $x^3-3x^2+3x-1=0.$

(*c*) $x^4-x^2+x-3=0.$
Ans. $x^4-2x^3-5x^2+5x+9=0.$

(*d*) $x^6-3x^4+3x^3-3x+2=0.$
Ans. $x^6-6x^5+9x^4-5x^3+6x^2-9x+4=0.$

(*e*) $x^2+x+1=0.$
Ans. $x^2+x+1=0.$

2. If α β, γ, δ be the roots of the biquadratic

$$x^4-px^3+qx^2-rx+s=0,$$

form an equation whose roots shall be $\alpha^2, \beta^2, \gamma^2, \delta^2$.

Hence find the values of $\Sigma\alpha^2$ and $\Sigma\alpha^2\beta^2\gamma^2$.

§ 34 Cube Roots of Unity.

The equation $x^3=1$ being of the third degree, has three roots. These are the three cube roots of unity.

We have $x^3-1\equiv(x-1)(x^2+x+1)=0$.

The equation $x-1=0$ gives unity as one of the three cube roots.

The other two are obtained by solving the quadratic

$$x^2+x+1=0.$$

These are $\dfrac{-1\pm i\sqrt{3}}{2}$ and are imaginary.

If one of the imaginary cube-roots of unity be denoted by w, the other is found to be w^2. Thus $1, w, w^2$ are the three cube roots of unity.

The sum of the three cube roots of unity is zero.

Moreover, w being a root of the equation $x^3=1$, $w^3=1$.

As an illustration of the use of the two results proved above, we shall find the continued product of

$$P+Q+R,\ P+wQ+w^2R, \text{ and } P+w^2Q+wR.$$

We have

$$\begin{aligned}&(P+wQ+w^2R)(P+w^2Q+wR)\\ &=P^2+w^3Q^2+w^3R^2+(w+w^2)PQ+(w^4+w^2)QR+(w+w^2)PR\\ &=P^2+Q^2+R^2-PQ-QR-PR, \text{ since } w^3=1 \text{ and } w+w^2=-1.\end{aligned}$$

Multiplying both sides by $P+Q+R$, we get

$$(P+Q+R)(P+w^2Q+wR)(P+wQ+w^2R)=P^3+Q^3+R^3-3PQR.$$

Ex. Show that $P^3+Q^3=(P+Q)(P+wQ)(P+w^2Q)$
and $P^3-Q^3=(P-Q)(P-wQ)(P-w^2Q)$.

§ 35. *To form an equation whose roots shall be the cubes of those of a given equation.*

Let the given equation $\sum\limits_{r=0}^{n} p_r x^{n-r}=0$, whose roots are $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots, \alpha_n$ be written as

$$\begin{aligned}(p_n+p_{n-3}x^3+p_{n-6}x^6+\ldots)&+x(p_{n-1}+p_{n-4}x^3+p_{n-7}x^6+\ldots)\\ &+x^2(p_{n-2}+p_{n-5}x^3+p_{n-8}x^6+\ldots\ldots)=0.\end{aligned}$$

i.e. in the form $P+xQ+x^2R=0$, P,Q,R being functions of x^3.

Then $$P+xQ+x^2R \equiv p_0 \prod_{r=1}^{n} (x-\alpha_r). \qquad (i)$$

Changing x into wx and w^2x in this identity, we get

$$P+wxQ+w^2x^2R \equiv p_0 \prod_{r=1}^{n} (wx-\alpha_r), \qquad (ii)$$

and $$P+w^2xQ+wx^2R \equiv p_0 \prod_{r=1}^{n} (w^2x-\alpha_r)\;; \qquad (iii)$$

P,Q,R being functions of x^3 remain unaltered.
Multiplying together (*i*), (*ii*) and (*iii*), we obtain

$$P^3+x^3Q^3+x^6R^3-3x^3PQR \equiv p_0^3 \prod_{r=1}^{n} (x^3-\alpha_r^3). \qquad (iv)$$

Replacing x^3 by x in (*iv*), we finally obtain

$$(p_n+p_{n-3}x+p_{n-6}x^2+\ldots)^3+x(p_{n-1}+p_{n-4}x+p_{n-7}x^2+\ldots)^3$$
$$+x^2(p_{n-2}+p_{n-5}x+p_{n-8}x^2+\ldots)^3-3x(p_n+p_{n-3}x+p_{n-6}x^2+\ldots)$$
$$(p_{n-1}+p_{n-4}x+p_{n-7}x^2+\ldots)(p_{n-2}+p_{n-5}x+p_{n-8}x^2+\ldots)$$
$$\equiv p_0^3 \prod_{r=1}^{n} (x-\alpha_r^3). \qquad (v)$$

The right hand side of this identity vanishes when
$x=\alpha_r^3$, $r=1, 2, 3, \ldots\ldots, n$.

The required equation is, therefore, obtained by equating to zero the left hand side of (*v*).

Example. Form an equation whose roots shall be the cubes of those of the equation :

$$2x^5+3x^4+4x^3-2x^2-3x+1=0.$$

Here $P=4x^3+1$, $Q=3x^3-3$ and $R=2x^3-2$.

Therefore, the required equation is

$$(1+4x)^3+x(3x-3)^3+x^2(2x-2)^3-3x(4x+1)(3x-3)(2x-2)=0\;;$$

or $$8x^5-69x^4+133x^3+85x^2-33x+1=0.$$

EXERCISES XXIII

Form equations whose roots shall be the cubes of the roots of :—

1. $x^4-x^3+2x^2+3x+1=0.$

Ans. $x^4+14x^3+50x^2+6x+1=0.$

2. $x^3+3x^2+2=0$. **Ans.** $x^3+33x^2+12x+8=0$.

3. $x^3+ax^2+bx+ab=0$.
Ans. $x^3+a^3x^2+b^3x+a^3b^3=0$.

4. $x^5+x^4+x^3+x^2+x+1=0$.
Ans. $x^5+x^4-2x^3-2x^2+x+1=0$.

§ 36. *To increase or decrease the roots of a given equation by an assigned quantity.*

Let $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_n$, be the roots of the equation

$$f(x) \equiv \sum_{r=0}^{n} p_r x^{n-r} = 0,$$

so that
$$f(x) \equiv p_0 \prod_{r=1}^{n} (x-\alpha_r). \qquad (i)$$

In this identity, change x into $(x+h)$, then

$$f(x+h) \equiv p_0 \prod_{r=1}^{n} (x+h-\alpha_r). \qquad (ii)$$

The right hand side of (ii) vanishes when

$$x=\alpha_r-h \; ; \; r=1, 2, 3 \ldots\ldots n.$$

Therefore $f(x+h)=0$, is the equation whose roots are each less by h than the roots of the equation $f(x)=0$.

Again changing x into $(x-h)$ in (i), we get

$$f(x-h)=0$$

as the equation whose roots are in excess of the roots of the given equation $f(x)=0$ by h. $f(x+h)$ and $f(x-h)$ are most easily calculated by Horner's Contracted Process given in Chapter I.

Example. Diminish by 2 the roots of the equation

$$f(x) \equiv 2x^5-2x^4+3x^3-2x^2-x-5=0.$$

We have to calculate $f(x+2)$ in this case.

Employing Horner's process,

$$
\begin{array}{r|l}
 & 2-\ 2+\ 3-\ 2-\ 1-\ 5 \\
2 & \underline{\ \ +\ 4+\ 4+14+\ 24+46} \\
 & 2+\ 2+\ 7+12+\ 23\ +41 \\
2 & \underline{\ \ +\ 4+12+38+100} \\
 & 2+\ 6+19+50\ +123 \\
2 & \underline{\ \ +\ 4+20+78} \\
 & 2+10+39\ +128 \\
2 & \underline{\ \ +\ 4+28} \\
 & 2+14\ +67 \\
2 & \underline{\ \ +\ 4} \\
 & 2+18
\end{array}
$$

we get $2x^5+18x^4+67x^3+128x^2+123x+41=0$ as the required equation.

EXERCISES XXIV

1. Find the equations whose roots are those of the equation $2x^5+x^4-3x^2-x+2=0.$

(i) each diminished by 2, (ii) each increased by 4.

(iii) each diminished by 1, (iv) each increased by 3.

Ans. (i) $2x^5+21x^4+88x^3+181x^3+179x+68=0$

(ii) $2x^5-39x^4+304x^3-1187x^2+2327x-1834=0.$

(iii) $2x^5+11x^4+24x^3+23x^2+7x+1=0.$

(iv) $2x^5-29x^4+168x^3-489x^2+719x-427=0.$

2. Diminish by 3, each of the roots of the equation :

$$x^5-4x^4+3x^2-4x+4=0.$$

Ans. $x^5+11x^3+42x^3+57x^2-13x-60=0.$

3. Increase by 3/2 each of the roots of the equation :

$$x^3-\frac{x}{4}-\frac{3}{4}=0 \qquad \textbf{Ans. } 4x^3-18x^2+26x-15=0.$$

§ 37. *To remove an assigned term from an equation.*

Let $\Phi_n(x)\equiv\sum_{r=0}^{n}{}^nC_r p_r x^{n-r}=0$ be the given equation.

Diminishing the roots of this equation by h, we obtain the equation

$$\phi_n(x+h)\equiv\sum_{r=0}^{n}{}^{n}C_r\phi_r(h)x^{n-r}=0,$$

where
$$\phi_r(h)\equiv\sum_{r=0}^{r}{}^{r}C_k p_k h^{r-k}.$$

To remove any term, say the $(r+1)$th, we equate to zero the coefficient of that term. We thus get $\phi_r(h)=0$.

This being an equation of the rth degree in (h), gives, in general, r values for h. To remove the said term, we diminish the roots of the given equation by any one of these r values. In particular, when it is the second term that has to be removed, we have

$$\phi_1(h)\equiv p_0h+p_1=0,\quad i.e.\ h=-\frac{p_1}{p_0}.$$

Thus, we have to increase the roots by $\dfrac{p_1}{p_0}$ to remove the second term.

Example 1. Remove the second term from the equation
$$f(x)\equiv x^4+12x^3-10x^2+5x-4=0.$$
Here we have to increase the roots by $12/4$ *i. e.* 3.
Employing Horner's process to calculate $f(x-3)$,

$$\begin{array}{r|l}
 & 1+12-10+\ \ 5-\ \ 4 \\
-3 & \ \ \ -\ 3-27+111-348 \\ \hline
 & 1+\ 9-37+116\,|-352 \\
-3 & \ \ \ -\ 3-18+165 \\ \hline
 & 1+\ 6-55\,|+281 \\
-3 & \ \ \ -\ 3-\ 9 \\ \hline
 & 1+\ 3\,|-64 \\
-3 & \ \ \ -\ 3 \\ \hline
 & 1+\ 0
\end{array}$$

$h = \dfrac{\text{Sum of roots}}{\text{degree}}$

$= -\dfrac{b}{a}$

we get the equation
$$x^4-64x^2+281x-352=0.$$

Example 2. Remove the third term from the equation :

$$x^5-10x^4+30x^3+5x-6=0.$$

Here we put $\phi_2(h)$ or $p_0h^2+2p_1h+p_2$

i.e. $\qquad h^2-4h+3$ equal to zero.

This gives $h=1, 3$.

To remove the third term therefore, we must diminish the roots of the given equation by either 1 or 3. Thus

$$\begin{array}{c|llllll}
 & 1 & -10 & +30 & +0 & +5 & -6 \\
1 & & +1 & -9 & +21 & +21 & +26 \\ \hline
 & 1 & -9 & +21 & +21 & +26 & \vert +20 \\
1 & & +1 & -8 & +13 & +34 & \\ \hline
 & 1 & -8 & +13 & +34 & \vert +60 & \\
1 & & +1 & -7 & +6 & & \\ \hline
 & 1 & -7 & +6 & \vert +40 & & \\
1 & & +1 & -6 & & & \\ \hline
 & 1 & -6 & \vert +0 & & & \\
1 & & +1 & & & & \\ \hline
 & 1 & -5 & & & &
\end{array}
\qquad
\begin{array}{c|llllll}
 & 1 & -10 & +30 & +0 & +5 & -6 \\
3 & & +3 & -21 & +27 & +81 & +258 \\ \hline
 & 1 & -7 & +9 & +27 & +86 & \vert +252 \\
3 & & +3 & -12 & -9 & +54 & \\ \hline
 & 1 & -4 & -3 & +18 & \vert +140 & \\
3 & & +3 & -3 & -18 & & \\ \hline
 & 1 & -1 & -6 & \vert +0 & & \\
3 & & +3 & +6 & & & \\ \hline
 & 1 & +2 & \vert +0 & & & \\
3 & & +3 & & & & \\ \hline
 & 1 & +5 & & & &
\end{array}$$

diminishing the roots by 1, we get the equation

$$x^5-5x^4+40x^2+60x+20=0\ ;$$

and diminishing the roots by 3, we get the equation

$$x^5+5x^4+140x+252=0.$$

EXERCISES XXV

Remove the second term from the following equations :—

1. $x^6-12x^5+3x^2-17x+300=0.$
   Ans. $x^6-60x^4-320x^3-717x^2-773x-42=0.$

2. $x^4+8x^3+x-5=0$
   Ans. $x^4-24x^2+65x-55=0.$

3. $x^6+6x^5+7=0.$
   Ans. $x^6-15x^4+40x^3-45x^2+24x+2=0.$

Remove the third term from each of the following equations :—

4. $x^4-4x^3-18x^2-3x+2=0.$

Ans. $x^4+8x^3-111x-196=0$ or $x^4-8x^3+17x-8=0.$

5. $2x^3-15x^2+24x-7=0.$

Ans. $2x^3+9x^2-23=0$ or $2x^3-9x^2+4=0.$

6. Remove the second term from the cubic equation
$$p_0x^3+3p_1x^2+3p_2x+p_3=0.$$

7. Remove the second term from the biquadratic equation
$$p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0.$$

8. If $\phi_n(x)\equiv p_0x^n+{}^nC_1p_1x^{n-1}+{}^nC_2p_2x^{n-2}+\dots\dots\dots$
$$\dots+{}^nC_{n-1}p_{n-1}x+{}^nC_np_n,$$
find the condition that the same transformation may remove from the equation $\phi_n(x)=0$,

(*i*) the second and third terms ; **Ans.** $p_0p_2-p_1^2=0.$

(*ii*) the second and fourth terms ; **Ans.** $p_0^2p_3-3p_0p_1p_2+2p_1^3=0.$

(*iii*) the second and fifth terms.

Ans. $p_0^3p_4-4p_0^2p_1p_3+6p_0p_1^2p_2-3p_1^4=0.$

9. Remove the second term from the equation
$$x^3+6x^2+12x-19=0$$
and hence solve it. **Ans.** $1, -2+3w, -2+3w^2.$

10. Remove the second term from the equations

(*i*) $x^4+16x^3+72x^2+64x-129=0,$

and (*ii*) $x^4+20x^3+143x^2+430x+462=0,$
and hence solve them.

Ans. (*i*) $-4\pm\sqrt{[12\pm\sqrt{145}]}$. (*ii*) $-7, -3, -5\pm\sqrt{3}$.

§ 38. Transformation in General.

If α be a root of the equation $f_n(x)=0$, then to form an equation whose corresponding root shall be $\phi(\alpha)$, we have simply to eliminate α between the equations

$$f_n(\alpha)=0 \text{ and } x=\phi(\alpha).$$

The equation in x so obtained is the required equation.

To get the new equation, we have first to connect the corresponding roots α, α' by a relation of the form $\alpha'=\phi(\alpha)$.

The relations between the roots and co-efficients of an equation set down in Chapter II are of much use in establishing such a connection between the corresponding roots.

A few examples will illustrate the method of procedure.

Example 1. If α, β, γ be the roots of the cubic

$$x^3+3qx+r=0,$$

form an equation whose roots shall be $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$.

We have, $(\beta-\gamma)^2=\beta^2+\gamma^2-2\beta\gamma$

$$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}.$$

$$=-6q-\alpha^2+\frac{2r}{\alpha}.$$

The required equation is obtained, therefore, by eliminating α between the equations :

$$\alpha^3+3q\alpha+r=0\,; \qquad (i)$$

and

$$x=-6q-\alpha^2+\frac{2r}{\alpha},$$

or

$$\alpha^3+(x+6q)\alpha-2r=0. \qquad (ii)$$

Subtracting (i) from (ii), we get

$$(x+3q)\alpha=3r \text{ or } \alpha=\frac{3r}{x+3q}.$$

Substituting this value of α in (i), we get

$$27r^3+3q(x+3q)^2.3r+r(x+3q)^3=0,$$

or

$$x^3+18qx^2+81q^2x+27(4q^3+r^2)=0.$$

This equation is called the 'Equation of Squared Differences' of the cubic $x^3+3qx+r=0$, and shall be of immense use to us later on.

Example 2. If α, β, γ be the roots of the equation
$$x^3-7x+6=0,$$
form an equation whose roots shall be
$$\alpha^2+3\alpha+2,\ \beta^2+3\beta+2,\ \gamma^2+3\gamma+2.$$

Here, we have to eliminate α between the equations
$$\alpha^3-7\alpha+6=0, \qquad (i)$$
and
$$\alpha^2+3\alpha+2=x \text{ or } \alpha^2+3\alpha+(2-x)=0 \qquad (ii)$$

Multiplying (ii) by α and substracting (i) from it, we get
$$3\alpha^2+(9-x)\alpha-6=0. \qquad (iii)$$

From (ii) and (iii), we have
$$\frac{\alpha^2}{-18-(2-x)(9-x)}=\frac{\alpha}{3(2-x)+6}=\frac{1}{(9-x)-9};$$
so that
$$(12-3x)^2=-x(-x^2+11x-36).$$

On simplification, the transformed equation is
$$x^3-20x^2+108x-144=0.$$

Example 3. a, b, c are the roots of the cubic :
$$x^3+px^2+qx+r=0\ ;$$
form an equation whose roots shall be $\dfrac{b}{c}+\dfrac{c}{b}, \dfrac{c}{a}+\dfrac{a}{c}, \dfrac{a}{b}+\dfrac{b}{a}$.

We have $a+b+c=-p$, $ab+bc+ca=q$ and $abc=-r$.

Therefore $$\frac{b}{c}+\frac{c}{b}=\frac{b^2+c^2}{bc}=\frac{(a+b+c)^2-2(ab+bc+ca)-a^2}{abc/a}$$
$$=\frac{a(p^2-2q-a^2)}{-r}.$$

Hence, the required equation will be obtained by eliminating a between the equations :
$$a^3+pa^2+qa+r=0, \qquad (i)$$
and
$$x=\frac{a(a^2+2q-p^2)}{r} \text{ or } a^3+(2q-p^2)a-rx=0. \qquad (ii)$$

Subtracting (*ii*) from (*i*), we get

$$pa^2+(p^2-q)a+r(1+x)=0. \qquad (iii)$$

From (*i*) and (*iii*), we have

$$qa^2+(pq-r-rx)a+rp=0 \qquad (iv)$$

From (*iii*) and (*iv*), we obtain

$$\frac{a^2}{rp(p^2-q)-r(1+x)(pq-r-rx)}=\frac{a}{rq(1+x)-rp^2}$$
$$=\frac{1}{p(pq-r-rx)-q(p^2-q)};$$

hence

$$[rq(1+x)-rp^2]^2=[rp(p^2-q)-r(1+x)(pq-r-rx)]$$
$$[p(pq-r-rx)-q(p^2-q)].$$

On simplification, this reduces to

$$r^2x^3+(3r-pq)rx^2+(3r^2-5pqr+rp^3+q^3)x$$
$$+(r^2-4pqr+2rp^3-p^2q^2+2q^3)=0.$$

which is the required equation.

EXERCISES XXVI

If α β, γ be the roots of the cubic $x^3+px^3+qx+r=0$, form equations whose roots shall be

1. $\dfrac{\beta+\gamma}{\alpha^2}, \dfrac{\gamma+\alpha}{\beta^2}, \dfrac{\alpha+\beta}{\gamma^2}$. 2. $\beta^2\gamma^2, \gamma^2\alpha^2, \alpha^2\beta^2$.

3. $\alpha(\beta+\gamma), \beta(\gamma+\alpha), \gamma(\alpha+\beta)$. 4. $\alpha^3, \beta^3, \gamma^3$.

5. $\dfrac{k}{\alpha}, \dfrac{k}{\beta}, \dfrac{k}{\gamma}$. 6. $\beta\gamma+\dfrac{1}{\alpha}, \gamma\alpha+\dfrac{1}{\beta}, \alpha\beta+\dfrac{1}{\gamma}$.

Ans. 1. $r^2x^3+(pq^2-qr-2rp^2)x^2+(p^4+4pr-3p^2q)x+pq-r=0$.

2. $x^3+(2pr-q^2)x^3+(p^2r^2-2qr^2)x-r^4=0$.

3. $x^3-2qx^2+q^2x+r^2+pr(x-q)=0$.

4. $x^3+(p^3-3pq+3r)x^2+(3r^2+q^3-3pqr)x+r^3=0$.

5. $rx^3+qkx^2+pk^2x+k^3=0$.

6. $rx^3+q(1-r)x^2+p(1-r)^2x+(1-r)^3=0$.

7. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$x^4-3x^3+2x^2-3x+1=0$$

form the equation whose roots shall be $\alpha^3+\alpha^2+\alpha+1$ etc.

Ans. $x^4-30x^3+45x^2=0$.

8. If α, β, γ be the roots of the cubic

$$p_0x^3+3p_1x^2+3p_2x+p_3=0,$$

form the equation whose roots are $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$.

9. Form the equation of squared differences of

(i) $x^3-7x+6=0$. Ans. $x^3-42x^2+441x-400=0$.

(ii) $x^3+6x^2+9x+4=0$. Ans. $x^3-18x^2+81x=0$.

10. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic

$$p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0,$$

form the sextic whose roots are

$$\beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta,\ \alpha+\delta,\ \beta+\delta,\ \gamma+\delta.$$

§ 39. A General Method* for finding equations of 'Squared Differences' and 'Semi-Sums'.

Let α, β be any two roots of the equation $f(x)=0$.

Choose h and k so that $\alpha=h+k$ and $\beta=h-k$.

We then have $f(h+k)=0$ and $f(h-k)=0$.

Expanding $f(h+k)$ and $f(h-k)$ by Taylor's Theorem, we get

$$f(h)+f'(h)k+f''(h).\ \frac{k^2}{2!}+\ldots+f^{(n)}(h)\ \frac{k^n}{n!}=0. \qquad (i)$$

and $$f(h)-f'(h)k+f''(h).\ \frac{k^2}{2!}-\ldots.+(-1)^nf^{(n)}(h)\frac{k^n}{n!}=0. \qquad (ii)$$

Hence, by addition and subtraction, from (i) and (ii) we have

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*This method is due to Professor Biswas.

$$f(h)+f''(h)\,\frac{k^2}{2!}+f\ (h).\,\frac{k^4}{4!}+\ldots\ldots\ldots=0. \qquad (iii)$$

and $$f'(h)+f'''(h).\frac{k^2}{3!}+f\,(h).\,\frac{k^4}{5!}+\ldots\ldots\ldots=0. \qquad (iv)$$

Eliminating h between (*iii*) and (*iv*), we shall get an equation in k^2.

Since $k=\frac{\alpha-\beta}{2}$ *i.e.* $k^2=\frac{(\alpha-\beta)^2}{4}$, replacing k^2 by $\frac{x}{4}$, we shall get the equation of the 'squared differences' of the roots of the given equation.

Again eliminating k between (*iii*) and (*iv*), we shall get an equation in h. Since $h=\frac{\alpha+\beta}{2}$, this equation shall be the equation of the 'Semi-Sums' of the roots of the equation $f(x)=0$.

The equation $f(x)=0$ shall have equal roots, if the equation of 'squared differences' has a zero root. The same equation shall have roots equal in magnitude but opposite in sign, if the equation of 'semi-sums' has a zero root.

Example 1. Form the equations of 'squared differences' and 'semi-sums' of the cubic $x^3+qx+r=0$.

Here $f(x)=x^3+qx+r$.

Therefore
$f(h)=h^3+qh+r,\ f'(h)=3h^2+q,\ f''(h)=6h$ and $f'''(h)=6$.

We thus have the equations

$$(h^3+qh+r)+\frac{k^2}{2}\,6h=0 \text{ and } (3h^2+q)+\frac{k^2}{6}.\,6=0\,;$$

or $$h^3+(q+3k^2)h+r=0, \qquad (i)$$

and $$3h^2+(q+k^2)=0. \qquad (ii)$$

From (*i*), we have $r^2=h^2[h^2+3k^2+q]^2$.

From (*ii*), we get $h^2=-\frac{k^2+q}{3}$.

Hence $r^2 = -\frac{k^2+q}{3}\left[\frac{8k^2+2q}{3}\right]^2$

Replacing k^2 by $\frac{x}{4}$, we get

$$27r^2 + (x+4q)(x+q)^2 = 0,$$

or $$x^3 + 6qx^2 + 9q^2x + 4q^3 + 27r^2 = 0,$$
which is the equation of the squared differences.

Again eliminating k^2 from (i) and (ii), we get

$$h^3 + qh + r - 3h(3h^2 + q) = 0,$$

or $$8h^3 + 2hq - r = 0.$$

The equation of the semi-sums of the roots of the cubic, therefore, is $8x^3 + 2qx - r = 0$.

N. B.—The equation whose roots are $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ can be obtained by multiplying the roots of $8x^3+2qx-r=0$ by 2.

Hence it is $\quad x^3 + qx - r = 0.$

Example 2. Form the equations of 'semi-sums' and 'squared differences' of the roots of the cubic

$$8x^3 - 12x^2 - 18x + 27 = 0.$$

Here $f(h) = 8h^3 - 12h^2 - 18h + 27$,
so that $\quad f'(h) = 24h^2 - 24h - 18$, $f''(h) = 48h - 24$ and $f'''(h) = 48$.

We have, thus, the two equations

$$(8h^3 - 12h^2 - 18h + 27) + \frac{k^2}{2}(48h - 24) = 0. \quad \ldots(i)$$

and $$(24h^2 - 24h - 18) + \frac{k^2}{6}.48 = 0. \quad \ldots(ii)$$

Eliminating k^2 between (i) and (ii), we get

$$2h^3 - 3h^2 = 0.$$

Therefore, the equation of semi-sums is $2x^3 - 3x^2 = 0$.

Eliminating h between (i) and (ii) *viz.* the equations

$$8h^3-12h^2+(24k^2-18)h+(27-12k^2)=0, \quad ...(iii)$$

and
$$12h^2-12h+(4k^2-9)=0, \quad ...(iv)$$

we get
$$16k^6-72k^4+81k^2=0.$$

Replacing k^2 by $\frac{x}{4}$, the equation of the squared differences is

$$x^3-18x^2+81x=0.$$

Ex. Form the equations of semi-sums and squared differences of the biquadratic $\quad x^4+px^2+qx+r=0.$

CHAPTER V

Location and Nature of the Roots of an Equation.

§ 40. Descarte's Rule of Signs.

The number of positive and negative roots of the equation $f_n(x)=0$ cannot exceed the number of changes of sign in $f_n(x)$ and $f_n(-x)$ respectively.

Let the signs of terms in the polynomial $f_n(x)$ be

$$+ + - - - + - + -$$

We show that multiplication by a binomial whose signs are $+ -$ introduces at least one more change of sign.

Writing only the signs of the several terms in the multiplication, we have

$$\begin{array}{cccccccccc}
+ & + & - & - & - & + & - & + & - & \\
+ & - & & & & & & & & \\
\hline
+ & + & - & - & - & + & - & + & - & \\
 & - & - & + & + & + & - & + & - & + \\
\hline
+ & \pm & - & \mp & \mp & + & - & + & - & +
\end{array}$$

Let us take the most unfavourable case *viz.*, that in which each ambiguity has been replaced by a continuation. We see that in the product, the signs before and after an ambiguity are unlike. Hence, whether we take the upper signs or the lower ones, the number of changes of sign will be the same. Taking the upper signs, the number of changes of sign in the product cannot be less than that in the arrangement

$$+ + - - - + - + - +$$

This series of signs is the same as the original one, with an additional change of sign at the end.

Thus, each factor $(x-\alpha)$ corresponding to a positive root α, introduces at least one change of sign.

Now suppose K roots of the equation $f_n(x)=0$, *viz.*, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_k$ are positive and the others are negative, zero or imaginary. Then

$$f_n(x) \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_k)F_{n-k}(x).$$

$F_{n-k}(x)$ may or may not have any changes of sign, but its multiplication by $(x-\alpha_1)$, $(x-\alpha_2)$, $(x-\alpha_3)\ldots\ldots,(x-\alpha_k)$ introduces in the product $f_n(x)$ at least k new changes of sign, so that $f_n(x)$ has at least k changes of sign.

Therefore, the number of positive roots of $f_n(x)=0$ cannot exceed the number of changes of sign in $f_n(x)$.

Again, the roots of the equation $f_n(-x)=0$ are equal in magnitude but opposite in sign to the roots of the equation $f_n(x)=0$. Therefore, the negative roots of $f_n(x)=0$ are the positive roots of $f_n(-x)=0$. Hence the number of negative roots of $f_n(x)=0$ cannot exceed the number of changes of sign in $f_n(-x)$.

It must be clearly understood that Descarte's Rule of signs does not give us the exact number of positive or negative roots of an equation, but gives only the superior limit for the number of such roots. Descarte's Rule helps us also, in proving the existence of imaginary roots. For if p be the greatest possible number of positive roots and q that of the negative roots of an equation of the nth degree, then the equation must have at least $n-(p+q)$ imaginary roots. Zero roots are readily found to start with.

Example. Show that the equation $x^6-3x^2-x+1=0$, has at least two imaginary roots.

Let $\qquad f(x)=x^6-3x^2-x+1,$

so that $\qquad f(-x)=x^6-3x^2+x+1.$

Now there are two changes of sign in $f(x)$ and two changes of sign in $f(-x)$. The equation $f(x)=0$ cannot, therefore, have more than two positive and more than two negative roots. There are no zero roots. Thus the given equation cannot have more than four real roots, therefore, it possesses at least two imaginary roots.

EXERCISES XXVII

1. Show that the equation $x^7-3x^4+12x^2+5x-4=0$ has at least two imaginary roots.

2. Show that if the signs of the terms of an equation be all positive, then the equation cannot have a positive root.

3. Show that if the signs of the terms of a complete equation be alternately positive and negative, then it cannot have a negative root.

§ 41. **Theorem.** *In any interval (a, b), a polynomial $f_n(x)$ with real co-efficients, changes continuously from $f_n(a)$ to $f_n(b)$.*

Let c and $(c+h)$ be any two values of x in the interval (a, b).

Then, by Taylor's Theorem, we have

$$f_n(c+h)=f_n(c)+\frac{h}{1\,!}f'_n(c)+\frac{h^2}{2\,!}f''_n(c)+\ldots+\frac{h^n}{n\,!}f_n^{(n)}(c)\;;$$

so that by taking h small enough, the difference between $f_n(c+h)$ and $f_n(c)$ can be made as small as we please. To a small change in the value of x, there corresponds, therefore, a small change in the value of $f_n(x)$. Hence, as x changes gradually from a to b, $f_n(x)$ changes from $f_n(a)$ to $f_n(b)$ passing through all the intermediate values.

§ 42. **Theorem.** *If $f_n(a)$ and $f_n(b)$ have opposite signs, then at least one root of $f_n(x)=0$ lies between a and b.*

Since $f_n(x)$ is a continuous function of x and it changes sign as x changes from a to b, it follows that $f_n(x)$ vanishes at least once in the interval (a, b). In other words, the equation $f_n(x)=0$ has at least one root between a and b.

But $f_n(x)$ may assume the value zero more than once in the interval (a, b). If $f_n(a)$ and $f_n(b)$ have opposite signs, $f_n(x)$ shall pass through zero an odd number of times in the interval (a, b). If $f_n(a)$ and $f_n(b)$ have the same sign then, either $f_n(x)$ does not pass through the value zero at all in the interval (a, b), or it passes through the value zero an even number of times.

Thus it is proved that if $f_n(a)$ and $f_n(b)$ have opposite signs, an odd number of roots of $f_n(x)=0$ lies between a and b; and that if $f_n(a)$ and $f_n(b)$ have the same sign, either no root or an even number of roots of $f_n(x)=0$ lies between a and b. Equal roots count according to their multiplicity.

Example. Show that the equation $x^4-12x^2+12x-3=0$ has a root between -4 and -3.

Let $f(x)=x^4-12x^2+12x-3$.

Then $f(-4)=13$,

and $f(-3)=-66$.

Thus $f(x)$ changes sign in the interval $(-4, -3)$.

The given equation has, therefore, a root between -4 and -3.

$$\begin{array}{r|l}
 & 1+0-12+12\quad-3 \\
-4 & \quad -4+16-16\quad+16 \\
\hline
 & 1-4+\ 4-\ 4 \parallel +13 \parallel
\end{array}$$

$$\begin{array}{r|l}
 & 1+0-12+12\quad-3 \\
-3 & \quad -3+\ 9+\ 9\quad-63 \\
\hline
 & 1-3-\ 3+21 \parallel -66 \parallel
\end{array}$$

Ex. 1. Show that the equation $x^4-12x^2+12x-3=0$ has a root between 2 and 3.

Ex. 2. Show that the equation $4x^3-13x^2-31x-275=0$ has a root between 6 and 7.

§ **43.** *Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, the co-efficient of the first term being positive.*

Consider the equation

$$f_n(x) \equiv x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0,$$

where n is odd.

We have $f_n(-\infty)=-\infty$, $f_n(o)=p_n$ and $f_n(+\infty)=+\infty$.

If p_n is positive, $f_n(x)$ changes sign in the interval $(-\infty, o)$ and the equation has a negative root.

If p_n is negative, $f_n(x)$ changes sign in the interval (o, ∞), and, therefore, $f_n(x)=0$ has a positive root.

This proves the theorem.

§ **44.** *Every equation of an even degree whose last term is negative and the co-efficient of the first term positive, has at least two real roots, one positive and one negative.*

Consider the equation

$$f_n(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0,$$

where n is even and p_n negative.

We have $f_n(-\infty) = +\infty$, $f_n(o) = p_n$ and $f_n(+\infty) = +\infty$.

Since p_n is negative. we see that $f_n(x)$ changes sign in the interval $(-\infty, o)$ as also in the interval (o, ∞).

The equation $f_n(x) = 0$, therefore, has at least two real roots, one between $-\infty$ and zero, which is negative ; and the other between zero and $+\infty$, which is positive.

§ 45. Search for Real Roots of an Equation.

Definitions.

(*i*) *Superior Limit of Positive Roots.* A number greater than each of the positive roots of an equation is called a superior limit of the positive roots.

(*ii*) *Inferior Limit of Positive Roots.* A number less than each of the positive roots of an equation is called an inferior limit of the positive roots.

(*iii*) *Superior Limit of Negative Roots.* A number greater than each of the negative roots of an equation is called a superior limit of the negative roots.

(*iv*) *Inferior Limit of Negative Roots.* A number less than each of the negative roots of an equation is called an inferior limit of the negative roots.

(*v*) *Superior Limit of Real Roots.* A number greater than each of the real roots of an equation is called a superior limit of the real roots.

(*vi*) *An inferior limit of real roots.* A number less than each of the real roots of an equation is called an inferior limit of real roots.

It is obvious that a superior limit of the positive roots and an inferior limit of the negative roots are respectively a superior and an inferior limit of the real roots of an equation.

§ 46. **Theorem.** *Any positive number h which renders the polynomial*

$$f_n(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_{n-1} x + p_n$$

and all its derived functions positive, is a superior limit of the positive roots of the equation $f_n(x)=0$.

Diminishing the roots of the equation $f_n(x)=0$ by h, we get the equation

$$f_n(x+h) \equiv x^n + \frac{f_n^{(n-1)}(h)}{(n-1)!} x^{n-1} + \frac{f_n^{(n-2)}(h)}{(n-2)!} x^{n-2} + \ldots + \frac{f'_n(h)}{1!} x + f_n(h) = 0.$$

If now h be such as to render the coefficients

$$f_n^{(r)}(h),\ r=0, 1, 2, 3, \ldots\ldots n-1$$

all positive, the equation $f_n(x+h)=0$ shall have no positive root.

The equation $f_n(x)=0$ can, therefore, have no root greater than h. Hence h is a superior limit of the positive roots of $f_n(x)=0$.

Note that if any value a of x renders positive each of the functions $\psi_n(x)$, $\psi'_n(x)$, $\psi''_n(x) \ldots\ldots$, $\psi_n^{(n)}(x)$, then any number $b>a$ shall make them positive also.

For we have

$$\psi_n^{(r)}(b) = \psi_n^{(r)}(a) + \frac{b-a}{1!} \psi_n^{(r+1)}(a) + \frac{(b-a)^2}{2!} \psi_n^{(r+2)}(a) + \ldots$$
$$\ldots + \frac{(b-a)^{n-r}}{(n-r)!} \psi_n^{(n)}(a)$$

Now each term on the right hand side being positive, $\psi_n^{(r)}(b)$ is necessarily positive.

The method of procedure will be clear from the following example.

Example. Find a superior limit of the positive roots of the equation $x^8+20x^7+4x^6+11x^5-12x^4-72x+28=0.$

Here $f(x)=x^8+20x^7+4x^6+11x^5-12x^4-72x+28.$

$f'(x)=8x^7+140x^6+24x^5+55x^4-48x^3-72.$

$$f''(x)=56x^6+840x^5+120x^4+220x^3-144x^2,$$
$$f'''(x)=336x^5+4200x^4+480x^3+660x^2-288x,$$
$$f^{iv}(x)=1680x^4+16800x^3+1440x^2+1320x-288,$$
$$f^{v}(x)=6720x^3+50400x^2+2880x+1320 \text{ ; and so on.}$$

Now $f^{v}(x)$ and all the succeeding derivatives are positive when $x>0$; $f^{iv}(x)$ is positive and so also are $f'''(x)$, $f''(x)$ and $f'(x)$ positive when $x>1$. $f(x)$ is positive when $x>2$. Therefore 2 is a superior limit of the positive roots of the given equation.

§ 47. As the roots of $f_n(-x)=0$ are equal in magnitude but opposite in sign to those of $f_n(x)=0$, an inferior limit of the negative roots of $f_n(x)=0$ is a superior limit (with its sign changed) of the positive roots of $f_n(-x)=0$.

As the roots of

$$x^n f_n\left(\frac{1}{x}\right)=0$$

are reciprocals of the roots of $f_n(x)=0$, an inferior limit of the positive roots of $f_n(x)=0$ is the reciprocal of a superior limit of the positive roots of

$$x^n f_n\left(\frac{1}{x}\right)=0.$$

A superior limit of negative roots of $f(x)=0$ is an inferior limit with its sign changed of the positive roots of $f(-x)=0$.

EXERCISES XXVIII

Find the superior and inferior limits of the positive and negative roots of the following equations :

1. $x^7-x^6+x^5+x^4-3x^3+4x^2-x-1=0.$
2. $x^5+4x^4+2x^3-18x^2-36x-72=0.$
3. $x^4-2x^3-2x^2-15x-3=0.$
4. $x^3-4x^2-5x+2=0.$

§ 48. A note on the separation of the roots of an equation.

Suppose $\psi(x)=0$ is the equation of 'squared differences' of th e roots of the equation $f(x)=0$.

Let k^2 be an inferior limit of the positive roots of the equation $\psi(x)=0$ and s a superior limit of the positive roots of $f(x)=0$. Then the difference between any two roots of $f(x)=0$ is necessarily greater than k, so that between any two numbers l and $l+k$ only one root of $f(x)=0$ can possibly lie. To find if a root of $f(x)=0$ does actually lie between l and $l+k$, we examine, whether or not $f(x)$ changes sign as x changes from l to $l+k$. If it does change sign there is one and only one root of $f(x)=0$ between l and $l+k$. If it does not change sign there is no root of $f(x)=0$ between l and $l+k$. If $f(l)$ or $f(l+k)$ vanishes l or $l+k$ is itself a root. In view of these observations, we try in particular the series of numbers

$$s,\ s-k,\ s-2k,\ s-3k, \ldots\ldots,\ t+k,\ t;$$

t being the inferior limit of the negative roots of $f(x)=0$. When k turns out to be small in magnitude, this method is useless in practice.

Example. Separate the roots of the equation

$$f(x)=x^3-3x^2-2x+5=0.$$

To obtain the equation of squared differences of the roots of the equation $x^3-3x^2-2x+5=0$, we have to eliminate h between the equations

$$(h^3-3h^2-2h+5)+\frac{k^2}{2\,!}(6h-6)=0, \tag{i}$$

and

$$(3h^2-6h-2)+\frac{k^2}{3\,!}(6)=0, \tag{ii}$$

and replace k^2 by $\dfrac{z}{4}$. We thus get the cubic

$$z^3-30z^2+225z-473=0. \tag{iii}$$

The equation whose roots are the reciprocals of those of (*iii*) is

$$F(u)\equiv 473u^3-225u^2+30u-1=0 \tag{iv}$$

We have $F'(u)=1419u^2-450u+30$,

and $F''(u)=2838u-450$.

Now, $F''(u)$ is positive when u is $>\frac{1}{6}$;

$F'(u)$ is positive when u is $>\frac{1}{4}$;

and $F(u)$ is positive when u is $>\frac{1}{3}$.

Hence 3 is an inferior limit of the positive roots of (iii).

Moreover $f(x)=x^3-3x^2-2x+5,$

$$f'(x)=3x^2-6x-2,$$

and $$f''(x)=6x-6,$$

Hence $f''(x)>0$ if $x>1,$

$$f'(x)>0 \text{ if } x>3,$$

and $$f(x)>0 \text{ if } x>3{\cdot}5.$$

Thus $3{\cdot}5$ is a superior limit of the roots of $f(x)=0$.

Again $f(-x)=x^3+3x^2-2x-5,$

$$f'(-x)=3x^2+6x-2,$$

and $$f''(-x)=6x+6.$$

$f''(-x), f'(-x)$ and $f(-x)$ are all positive when $x>1{\cdot}5$.

Therefore, $-1{\cdot}5$ is an inferior limit of the real roots of $f(x)=0$.

We have, thus, to try the series of numbers

$$3{\cdot}5,\ 1{\cdot}8,\ 0{\cdot}1 \text{ and } -1{\cdot}6.$$

On substitution, we find that

$$f(3{\cdot}5)=+\text{ive}, f(1{\cdot}8)=-\text{ive}, f(0{\cdot}1)=+\text{ive}$$

and $$f(-1{\cdot}6)=-\text{ ive}.$$

Therefore the roots of $f(x)=0$ lie in the intervals

$$(-1{\cdot}6,\ 0{\cdot}1),\ (0{\cdot}1,\ 1{\cdot}8) \text{ and } (1{\cdot}8,\ 3{\cdot}5).$$

On closer examination, the roots are found to lie in the intervals $(-1{\cdot}6, -1)$, $(1, 1{\cdot}8)$ and $(3, 3{\cdot}5)$.

Ex. Separate the real roots of the following equations

(i) $x^4-7x^2+18x-8=0$. **Ans.** $(-4, -3)$, $(0, 1)$.

(ii) $x^3-7x+7=0$. **Ans.** $(1{\cdot}7, 1{\cdot}6)$, $(1{\cdot}3, 1{\cdot}4)$, $(-4, -3)$.

§ 49. The Number of Distinct Real Roots of an Equation.

'Descartes Rule of Signs', while so handy in practice, does not help us in determining the exact number of the real roots of an equation. The methods given by Budan, Newton, Waring, Fourier, Sylvester, and others have the same defect. Sturm's method, which we shall now discuss, gives the exact number of the distinct real roots of an equation in any given interval. The method is rather laborious and is used only when the desired information cannot be obtained by the use of methods described in the foregoing sections of this chapter.

§ 50. Sturm's Functions.

Let $f(x)$ be any polynomial of degree n, and let $f_1(x)$ be its first derivative. Divide $f(x)$ by $f_1(x)$ and let the remainder with its sign changed be denoted by $f_2(x)$. Divide $f_1(x)$ by $f_2(x)$ and denote the remainder with its sign changed by $f_3(x)$ and so on. This process is similar to that of finding the H. C. F. of $f(x)$ and $f_1(x)$, with this modification, that the sign of the remainder is changed every time it is obtained. The successive remainders go on diminishing in degree till we reach finally a remainder which is either numerical or divides the remainder immediately preceding it exactly i.e., without leaving a remainder. The auxiliary functions $f_2(x), f_3(x), f_4(x)......f_m(x)$ along with $f(x)$ and $f_1(x)$ are called 'Sturm's Functions'. In the following discussion, for the sake of uniformity, we shall, sometimes, write $f_0(x)$ for $f(x)$.

From the method of construction of Sturm's Auxiliary Functions $f_2(x), f_3(x)$ etc., the following relations hold between them

$$
\begin{aligned}
f_0(x) &= q_1 f_1(x) - f_2(x),\\
f_1(x) &= q_2 f_2(x) - f_3(x),\\
f_2(x) &= q_3 f_3(x) - f_4(x),\\
\cdots\cdots &\cdots\cdots\cdots\cdots\\
f_{k-1}(x) &= q_k f_k(x) - f_{k+1}(x)\\
\cdots\cdots &\cdots\cdots\cdots\cdots\\
f_{m-2}(x) &= q_{m-1} f_{m-1}(x) - f_m(x),
\end{aligned}
$$

where the q's are the quotients obtained in the process of division explained above.

§ 51. In our proof of 'Sturm's Theorem', we shall need **Lemma.**—*If α be a root of the equation $f(x)=0$ and h a positive quantity so small that $f(x)=0$ has no root other than α in the interval $(\alpha-h,\ \alpha+h)$, then $f(x)$ and its first derivative $f'(x)$ have different signs when $x=\alpha-h$ and like signs when $x=\alpha+h$.*

We have $f(\alpha-h)=f(\alpha)-hf'(\alpha)+\dfrac{h^2}{2\,!}f''(\alpha)-\ldots\ldots,$

and $$f'(\alpha-h)=f'(\alpha)-hf''(\alpha)+\frac{h^2}{2\,!}f'''(\alpha)-\ldots\ldots$$

If $f'(\alpha)\neq 0$ as is the case when $f(x)=0$ has no equal roots, then since $f(\alpha)=0$, taking h sufficiently small, we find that $f(\alpha-h)$ and $f'(\alpha)$ have opposite signs while $f'(\alpha-h)$ and $f'(\alpha)$ have the same signs. Hence $f(\alpha-h)$ and $f'(\alpha-h)$ have different signs.

When α is a k-ple root of $f(x)=0$,

$f(\alpha), f'(\alpha), f''(\alpha),\ldots\ldots, f^{(k-1)}(\alpha)$ vanish and we have

$$f(\alpha-h)=(-1)^k\frac{h^k}{k\,!}f^{(k)}(\alpha)+(-1)^{k+1}\frac{h^{k+1}}{(k+1)\,!}f^{(k+1)}(\alpha)+\ldots\ldots$$

and $$f'(\alpha-h)=(-1)^{k-1}\frac{h^{k-1}}{(k-1)\,!}f^{(k)}(\alpha)+(-1)^k\frac{h^k}{k\,!}f^{(k+1)}(\alpha)+\ldots\ldots$$

whence it follows that $f(\alpha-h)$ and $f'(\alpha-h)$ have different signs. Similarly it can be shown that $f(\alpha+h)$ and $f'(\alpha+h)$ have like signs.

§ 52. *Sturm's Theorem states that the number of distinct real roots of $f(x)=0$ in the interval (a, b) where $a<b$, is equal to the excess of the number of changes of sign in the sequence*

$$f_0(x), f_1(x), f_2(x), f_3(x),\ldots\ldots f_m(x)$$

when $x=a$ over the number of changes of sign when $x=b$; a and b being arbitrary but such that none of the Sturm's function's $f_0(x), f_1(x), f_2(x),\ldots\ldots f_m(x)$ vanishes when $x=a$ or b.

The proof of Sturm's Theorem is divided into two parts.

Firstly, when $f(x)=0$ has no equal roots.

In this case, the last of Sturm's functions is numerical, $(\neq 0)$; for, otherwise $f(x)$ and $f_1(x)$ will have a common factor

involving x and the equation $f(x)=0$ will have equal roots. Moreover, no two consecutive f's can vanish for the same value of x, for, if they did, they and all their predecessors will have a common factor involving x, contradicting again the supposition that the equation $f(x)=0$ has no equal roots. Lastly, we observe that if any $f_k(x)$, $k>1$, vanishes when $x=\alpha$, then, for values of x in the immediate neighbourhood of α, $f_{k-1}(x)$ and $f_{k+1}(x)$ have opposite signs since for $x=\alpha$ they must be equal in magnitude but opposite in sign, as is apparent from the relation

$$f_{k-1}(x)=q_k f_k(x)-f_{k+1}(x)\,;\ k>1.$$

It may also be noted that the Sturm's Functions being all polynomials in x with real co-efficients, are continuous and any $f_k(x)$, $k>0$, can change sign only by passing through the value zero.

We show now, that the sequence

$$f_0(x), f_1(x), f_2(x), \ldots\ldots\ldots, f_m(x)$$

loses one change of sign as x varies continuously from $\beta-h$ to $\beta+h$, β being a root of the equation $f(x)=0$ and h a positive quantity so small that none of Sturm's functions excepting $f(x)$ vanishes in the interval $(\beta-h, \beta+h)$. We further suppose h to be so small that no root of $f(x)=0$ other than β lies in this interval.

From the Lemma of § 55 we know that $f(\beta-h)$ and $f_1(\beta-h)$ have opposite signs, while $f(\beta+h)$ and $f_1(\beta+h)$ have like signs. Moreover, no other member of the sequence changes sign in the interval $(\beta-h, \beta+h)$. Hence one and only one change of sign is lost as x increases continuously from $\beta-h$ to $\beta+h$.

Let us now examine if there is any gain or loss of a change of sign as x varies from $c-\varepsilon$ to $c+\varepsilon$ where c is a root of $f_k(x)=0$, $k\neq 0$, and ε a positive number so small that no Sturm's function other than $f_k(x)$ vanishes in the interval $(c-\varepsilon, c+\varepsilon)$. We have seen that in this case, $f_{k-1}(x)$ and $f_{k+1}(x)$ have opposite signs in the immediate neighbourhood of $x=c$. Hence whatever the signs of $f_k(x)$ may be when $x=c-\varepsilon$ and when $x=c+\varepsilon$, no change of sign is either gained or lost by the sequence.

Thus, it is proved that the sequence loses a change of sign only when x passes through a root of $f(x)=0$.

As x varies continuously from a to b, therefore, the sequence loses as many changes of sign as there are roots of the equation $f(x)=0$ in the interval (a, b).

If any of Sturm's functions vanish at the end points of the interval (a,b), we can consider the slightly different range $(a-h, b+h)$, h being a suitable small positive quantity.

Secondly, when $f(x)=0$ has equal roots.

Let H be the H.C.F. of $f(x)$ and its first derivative $f_1(x)$, then H divides each one of the Sturm's functions without a remainder.

Let $f_k(x) \equiv \text{H}.\ \text{F}_k(x),\ k=0, 1, 2, \ldots\ldots\ldots\ldots, m$;
then the equation $\text{F}_0(x)=0$ has the same roots as the equation $f(x)=0$, with this difference, that every root, multiple or otherwise of $f(x)=0$ occurs once and only once in the equation $\text{F}_0(x)=0$. Moreove $\text{F}_0(x)$ and $\text{F}_1(x)$ have no common factor involving x and no two consecutive members of the sequence

$$\text{F}_0(x),\ \text{F}_1(x),\ \text{F}_2(x) \ldots\ldots\ldots\ldots,\ \text{F}_m(x)$$

vanish for the same value of x. Corresponding to the relations of § 50, we now have

$$\text{F}_{k-1}(x)=q_k\text{F}_k(x)-\text{F}_{k+1}(x), \quad k=1, 2, 3, \ldots\ldots, m-1.$$

The number of changes of sign in the Sturm's sequence for any value of x for which no member of the sequence vanishes, is the same as the number of changes of sign in the sequence

$$\text{F}_0(x),\ \text{F}_1(x),\ \text{F}_2(x) \ldots\ldots\ldots,\ \text{F}_m(x).$$

Hence, proceeding as in the first part of the proof, it can be shown that the sequence $\text{F}_0(x)$, $\text{F}_1(x)$, $\text{F}_2(x), \ldots\ldots, \text{F}_m(x)$ loses no change of sign in passing through a root of any $\text{F}_k(x)=0$ other than $\text{F}_0(x)=0$.

To show that one change of sign is lost as x varies continuously from $\beta-h$ to $\beta+h$, where β is a root of $\text{F}_0(x)=0$ and h is a suitable small positive quantity, we notice that $f(x)$ and $f_1(x)$ have different signs when $x=\beta-h$, and like signs when $x=\beta+h$. Therefore, also $\text{F}_0(x)$ and $\text{F}_1(x)$ have diferent signs when $x=\beta-h$ and like signs when $x=\beta+h$ and the theorem follows as before.

§ 53. Since in Sturm's theorem, we are concerned only with the signs of the functions $f(x), f_1(x), f_2(x), \ldots\ldots, f_m(x)$, any of them can be multiplied by a positive constant or even by a polynomial in x which remains positive throughout the interval (a,b). In a similar manner, we can divide out also by positive numbers or expressions which remain positive in the interval (a,b). These observations are very useful in saving labour.

§ 54. We give some examples to clear up ideas.

Exampe 1. Find the number of distinct real roots of the equation $x^3-3x+1=0$ and locate them.

Sturm's Functions are calculated as follows :—

$$
\begin{array}{c|r|r|c}
x & f_0(x)=x^3-3x+1 & f_1(x)=3x^2-3 & \\
 & x^3-x & \text{Dividing by 3, we get} & \\
\cline{2-2}
 & -2x+1 & x^2-1 & \\
 & \text{changing sign} & 2 & \\
\cline{3-3}
 & f_2(x)=2x-1 & 2x^2 \quad -2 & x \\
 & & 2x^2-x & \\
\cline{3-3}
 & & x-2 & \\
 & & 2 & \\
\cline{3-3}
 & & 2x-4 & 1 \\
 & & 2x-1 & \\
\cline{3-3}
 & & -3 & \\
 & & \text{changing sign} & \\
 & & f_3(x)=3 &
\end{array}
$$

We have thus
$$
\begin{aligned}
f_0(x)&=x^3-3x+1.\\
f_1(x)&=x^2-1,\\
f_2(x)&=2x-1,\\
\text{and}\qquad f_3(x)&=3.
\end{aligned}
$$

For the calculation of the number of changes of sign lost, we use the following scheme :

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	changes of sign
$-\infty$	$-$	$+$	$-$	$+$	3
$+\infty$	$+$	$+$	$+$	$+$	0

This shows that the roots are all distinct and real.

To locate the roots, we give arbitrary values to x and find by trial where a change of sign is lost. The superior and inferior limits of the real roots of the equation may first be found

with advantage. In the present case these are 2 and -2 respectively. Using the above scheme, we now have

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	changes of sign
-2	$-$	$+$	$-$	$+$	3
$-1-e$	$+$	$+$	$-$	$+$	2
0	$+$	$-$	$-$	$+$	2
$1-e$	$-$	$-$	$+$	$+$	1
$1+e$	$-$	$+$	$+$	$+$	1
2	$+$	$+$	$+$	$+$	0

Thus the roots lie between $(-2, -1)$, $(0, 1)$, and $(1, 2)$.

Example 2. Find the number of real distinct roots of the equation $x^6-2x^5-4x^4+12x^3-3x^2-18x+18=0$, and locate them.

Sturm's functions are calculated as follows* :—

f_0	1−	2−	4+	12−	3−	18+	18
	6						
1	6−	12−	24+	72−	18−	108+	108
	6−	10−	16+	36−	6−	18	
	−	2−	8+	36−	12−	90+	108
		3					
−1	−	6−	24+	108−	36−	270+	324
	−	6+	10+	16−	36+	6+	18
		2)	−34+	92+	0−	276+	306
			−17+	46+	0−	138+	153
f_2			17−	46+	0+	138−	153
			7				
−17			119−	322+	0+	966−	1071
			119−	1734−	357+	5202	
				1412+	357−	4236−	1071
				7			
−1412				9884+	2499−	29652−	7497
				9884−	144024−	29652+	432072
				146523)	146523+	0−	439569
					1+	0−	3
f_4					−1+	0+	3

* For want of space, the left and right sides in the process are given separately.

$6-\ 10-\ 16+\ 36-\ 6-\ 18$	
$3-\ 5-\ 8+\ 18-\ 3-\ 9$	f_1
17	
$51-\ 85-136+\ 306-\ 51-\ 153$	3
$51-138+\ 0+\ 414-459$	
$53-136-\ 108+408-\ 153$	3
$51-138+\ 0+414-\ 459$	
$2+\ 2-\ 108-\ 6+\ 306$	
17	
$34+\ 34-1836-102+5202$	2
$34-\ 92+\ 0+276-\ 306$	
$18\ )\ 126-1836-378+5508$	
$7-\ 102-\ 21+\ 306$	
$-7+\ 102+\ 21-\ 306$	f_3
$-7+\ 0+\ 21$	7
$102+\ 0-\ 306$	-102
$102+\ 0-\ 306$	
0	

Thus $f_0(x)=x^6-2x^5-4x^4+12x^3-3x^2-18x+18$.

$f_1(x)=3x^5-5x^4-8x^3+18x^2-3x-9,$

$f_2(x)=17x^4-46x^3+138x-153,$

$f_3(x)=-7x^3+102x^2+21x-306,$

$f_4(x)=-x^2+3.$

This shows that $f(x)=0$ has equal roots. Proceeding as in example 1, we have, now, the following scheme of signs:—

x	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	changes
$-\infty$	+	−	+	+	−	3
-2	+	−	−	+	−	3
0	+	−	−	−	+	2
2	+	+	+	+	−	1
∞	+	+	+	−	−	1

The given equation has, therefore, only two distinct real roots and they lie in the intervals $(-2, 0)$ and $(0, 2)$.

Since the H.C.F. of $f(x)$ and $f_1(x)$ is x^2-3, the real roots of the equation are $-\sqrt{3}, -\sqrt{3}, \sqrt{3}, \sqrt{3}$.

The remaining two roots are imaginary.

EXERCISES XXIX

Determine the number of the distinct real roots of the following equations and locate them :—

1. $x^3-x^2-10x+1=0$.

   Ans. 3 ; $(-3, -2), (0, 1), (3, 4)$.

2. $x^5-5x^4+9x^3-9x^2+5x-1=0$.

   Ans. 3 ; $(0, 1-\varepsilon), (1-\varepsilon, 1+\varepsilon), (2, 3)$.

3. $x^4-2x^3-3x^2+10x-4=0$.

   Ans. 2 ; $(-3, -2), (0, 1)$.

4. $x^4-5x^3+9x^2-7x+2=0$.

   Ans. 2 ; $(0, 1+\varepsilon)(1+\varepsilon, 2+\varepsilon)$.

5. $x^5+2x^4+x^3-x^2-2x-1=0$.

   Ans. 2 ; $(0, 1+\varepsilon)\ (-1-\varepsilon, 0)$.

6. $x^5+3x^4+2x^3-3x^2-2x-2=0$.

   Ans. 1 ; $(1, 2)$.

7. $x^4-5x^2+8x-10=0$.

8. $2x^6-18x^5+60x^4-120x^3-30x^2+18x-5=0$.

Solution. The Sturm's functions are calculated as follows :—

$$
\begin{array}{r|l}
f & 2-18+60-120-30+18-5 \\
 & 2-15+40-60-10+3 \\
\hline
 & -3+20-60-20+15-5 \\
 & 2 \\
\hline
-3 & -6+40-120-40+30-10 \\
 & -6+45-120+180+30-9 \\
\hline
 & -5+0-220+0-1 \\
\hline
f_2 & 5+0+220+0+1
\end{array}
\qquad
\begin{array}{l}
6)\,12-90+240-360-60+18 \\
\hline
\quad 2-15+40-60-10+3 \;\; f'
\end{array}
$$

We notice that $f_2(x) \equiv 5x^4+220x^2+1$ is positive for all real values of x. Since division by a positive quantity is permissible in the calculation of Sturm's functions, we may take $f_2(x)=1$ and further calculation ends. The changes of sign are given by the following scheme :—

x	$f(x)$	$f_1(x)$	$f_2(x)$	changes
$-\infty$	$+$	$-$	$+$	2
-1	$+$	$-$	$+$	2
0	$-$	$+$	$+$	1
5	$-$	$+$	$+$	1
6	$+$	$+$	$+$	0
∞	$+$	$+$	$+$	0

Thus there are two real distinct roots, one between -1 and 0 and the other between 5 and 6.

9. $x^5-x^3+4x^2-3x+2=0$.

10. $8x^4+4x^3-18x^2+11x-2=0$.

§ 55. *To find the conditions that the roots of the equation $f(x)=0$ may be all real and distinct.*

Let the given equation be of the nth degree in x, and let $f(x), f_1(x), f_2(x), f_3(x)\ldots\ldots, f_m(x)$ be the sequence of Sturm's functions for the equation.

Since each of these functions is of a degree lower than its predecessor, the number of Sturm's functions cannot exceed $(n+1)$. If all the roots of $f(x)=0$ are real and distinct the sequence must lose n changes of sign as x varies from $-\infty$ to $+\infty$. It is, therefore, necessary that the number of Sturm's functions for the equation $f(x)=0$, be exactly $(n+1)$. Moreover, it is essential that for $x \to -\infty$, the signs of Sturm's functions should be alternately positive and negative and for $x \to +\infty$ there should be no change of sign in them. For these conditions to be satisfied, it is sufficient if the leading terms of Sturm's functions have coefficients of the same sign. It is easily shown that these two conditions are sufficient also.

When the sequence of Sturm's functions has $(n+1)$ members it shall be said to be 'complete'.

Example 1. There are k changes of sign among the coefficients of the leading terms of a complete Sturmian sequence, show that the equation has $2k$ imaginary roots.

We notice that when $x \to -\infty$ or $+\infty$, the signs of Sturm's functions depend upon the signs of the coefficients of their leading terms. If there are k changes among the signs of these coefficients, there shall be k changes of sign in the Sturmian sequence for $x \to +\infty$ and $(n-k)$ changes of sign when $x \to -\infty$ because in the latter case each continuity is replaced by a discontinuity and *vice versa*. Thus, the equation shall have $n-2k$ real roots all different and therefore $2k$ imaginary roots.

Example 2. If $f(x)$, $f_1(x)$, $f_2(x)$......,$f_k(x)$,......,$f_m(x)$ be Sturm's functions corresponding to the equation $f(x)=0$, show that the number of real distinct roots of $f_k(x)=0$ is not less than the number of changes of sign lost (or gained) by the sequence $f_k(x), f_{k+1}(x), f_{k+2}(x), \ldots\ldots, f_m(x)$ as x varies from $-\infty$ to $+\infty$.

Here, we consider the relations

$$\begin{aligned}
f_k(x)&=q_{k+1}f_{k+1}(x)-f_{k+2}(x),\\
f_{k+1}(x)&=q_{k+2}f_{k+2}(x)-f_{k+3}(x),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
f_{m-2}(x)&=q_{m-1}f_{m-1}(x)-f_m(x),
\end{aligned}$$

when $f(x)=0$ has no equal roots or the relations

$$\begin{aligned}
F_k(x)&=q_{k+1}F_{k+1}(x)-F_{k+2}(x),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
F_{m-2}(x)&=q_{m-1}F_{m-1}(x)-F_m(x),
\end{aligned}$$

where the F's are obtained by dividing the f's by H, the H.C.F. of $f(x)$ and $f_1(x)$, when $f(x)=0$ has equal roots.

Proceeding as in the proof of Sturm's theorem, it can be shown that the sequence $f_k(x), \ldots\ldots, f_m(x)$ can neither lose nor gain a change of sign as x passes through a root of any of the equations $f_t(x)=0$ or $F_t(x)=0$, $t=k+1, k+2, \ldots\ldots, m-1$.

A change of sign can be lost or gained only when x passes through a root of $f_k(x)=0$ or $F_k(x)=0$.

This proves the proposition.

As corollaries of the above, we have

(1) If $f_k(x)=0$ has no real root, then the sequence of Sturm's functions need not be prolonged beyond $f_k(x)$;

(2) If all the roots of $f(x)=0$ be real and distinct; then all the roots of $f_k(x)=0$ are also real and distinct.

EXERCISES XXX

1. If the equation $f_k(x)=0$ has q imaginary roots, show that the equation $f(x)=0$ has no less.

2. If the signs of the leading co-efficients in a quintic $f(x)$ and the Sturm's functions $f_2(x)$ and $f_3(x)$ be $+$, $-$ and $+$ respectively, find the number of real roots of the quintic.

3. If $f_k(x)=0$ has roots equal in pairs, show that Sturm's functions beyond $f_k(x)$ need not be considered.

4. Locate the roots of $f(x)\equiv x^4-5x^2+8x+10=0$ without finding Sturm's functions beyond $f_2(x)$.

[Hint. The equation $f_2(x)\equiv 5x^2-12x-20=0$ has two real roots α and β ; α between -2 and -1, and β between 3 and 4. Consider, therefore, the changes of sign lost by the sequence $f(x), f_1(x), f_2(x)$ in the intervals

$$(-\infty, \alpha-\varepsilon), (\alpha+\varepsilon, \beta-\varepsilon), (\beta+\varepsilon, \infty)$$

where ε is a suitable small positive number.

5. If functions be formed in the manner of Sturm's functions by using an arbitrary polynomial $\phi(x)$ in place of $f_1(x)$, the degree of $\phi(x)$ being less than that of $f(x)$, where does Sturm's method fail? Hence state the basic idea in the proof of Sturm's theorem.

6. If $f_k(x)$ is a complete polynomial with terms alternately positive and negative, show that in determining the negative roots of $f(x)=0$, Sturmians beyond $f_k(x)$ may be safely ignored.

State a similar result for positive roots of $f(x)=0$.

§ 56. Nature of the Roots of an Equation.

Sturm's Theorem can be used with advantage in determining the nature of the roots of an equation. Here we consider the Quadratic, Cubic and the Biquadratic.

(i) Let $f(x)=x^2+2bx+c=0$.

The Sturm's functions are $f(x)=x^2+2bx+c$,

$$f_1(x)=x+b$$

and $$f_2(x)=b^2-c.$$

The roots are both real and distinct, if b^2-c is positive, for then we have the following scheme of changes of sign :—

x	$f(x)$	$f_1(x)$	$f_2(x)$	changes
$-\infty$	$+$	$-$	$+$	2
$+\infty$	$+$	$+$	$+$	0

(ii) Consider now the cubic

$$x^3+3ax+b=0.$$

Sturm's functions, in this case, are

$$f(x)=x^3+3ax+b,$$

$$f_1(x)=x^2+a,$$

$$f_2(x)=-2ax-b,$$

$$f_3(x)=-4a^3-b^2.$$

[To find $f_3(x)$, before dividing $f_1(x)$ by $f_2(x)$, we multiply $f_1(x)$ by $4a^2$ which is a positive quantity].

The roots shall be all real and distinct if the coefficients of leading terms in $f(x)$, $f_1(x)$, $f_2(x)$ and $f_3(x)$ have the same sign ; *i.e.* if $2a$ and $(4a^3+b^2)$ are both negative.

Two roots shall be equal if $4a^3+b^2$ is equal to zero.

All the three roots shall be equal if $a=0$ and $4a^3+b^2=0$.

If $a>0$, $f_2(x)$ and $f_3(x)$ need not be taken into account, and we have the following scheme of changes of sign :—

x	$f(x)$	$f_1(x)$	changes
$-\infty$	$-$	$+$	1
0	b	$+$	?
∞	$+$	$+$	0

Thus, if a is positive, the cubic has only one real root which is positive if b is negative and negative if b is positive.

Ex. 1. What happens when $a<0$ and $4a^3+b^2>0$?

Ex. 2. Find the nature and signs of the real roots of the cubic when $a=0$.

Ex. 3. Find the conditions that the cubic may have two positive roots and one negative root.

Ans. $b<0, 4a^3+b^2<0$.

Ex. 4. Find the conditions that the quadratic

$$x^2+2bx+c=0,$$

may have two positive roots.

(iii) Let the biquadratic be put in the form

$$x^4+6Hx^2+4Gx+(I-3H^2)=0.$$

Then the Sturm's functions for the biquadratic are

$$f(x)=x^4+6Hx^2+4Gx+(I-3H^2),$$
$$f_1(x)=x^3+3Hx+G,$$
$$f_2(x)=-3Hx^2-3Gx-(I-3H^2),$$
$$f_3(x)=(IH-3G^2-12H^3)x-IG.$$

To find $f_4(x)$, we have to divide $f_2(x)$ by $f_3(x)$ and change the sign of the remainder. Now the remainder is the same as the value of $f_2(x)$ when $x=\dfrac{IG}{IH-3G^2-12H^3}$

For convenience, we write

$$IH-G^2-4H^3=J.$$

Then $IH-3G^2-12H^3=3J-2IH$.

Hence we may take $f^4(x)$

$$=3H\left(\frac{IG}{3J-2IH}\right)^2+3G\left(\frac{IG}{3J-2IH}\right)+I-3H^2$$

or multiplying by the positive number $(3J-2IH)^2$, we may take
$f_4(x)=3HI^2G^2+3IG^2(3J-2IH)+(I-3H^2)(3J-2IH)^2$

$$=9J^2(I-3H^2)-3IJ(4HI-12H^3-3G^2)-3HI^2G^2+4I^3H^2-12I^2H^4$$

$$=9J^2(I-3H^2)-3IJ(IH+3J)+I^2H(4IH-3G^2-12H^3)$$

$$=9J^2(I-3H^2)+I(IH+3J)(IH-3J)$$

$$=H^2(I^3-27J^2).$$

Since H^2 is positive, we can take finally

$$f_4(x)=I^3-27J^2.$$

All the Sturm's functions are thus found. We now proceed to consider the nature of the roots of the biquadratic.

(A) All roots real and distinct. The required conditions are $H<0$, $3J-2IH>0$ and $I^3-27J^2>0$.

(B) Two roots real and distinct. In this case, we have the scheme

x	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	changes
$-\infty$	$+$	$-$	$-H$	$-(3J-2IH)$	(I^3-27J^3)	
$+\infty$	$+$	$+$	$-H$	$3J-2IH$	(I^3-27J^2)	

There shall be three changes of sign in the first row and one in the second if $I^3-27J^2<0$
and either $H<0$ or $3J-2IH<0$.

(C) All the roots imaginary. In this case, we must have equal number of changes of sign in the two rows. Hence the required conditions are $I^3-27J^2>0$ and $H>0$; or $3J-2IH<0$.

(D) Two roots are equal when $I^3-27J^2=0$.

(E) Three roots are equal when $3J-2IH=0$ and $IG=0$; and $f_2(x)=0$ has equal roots ;

i.e. when $IG=0$, $3J=2IH$ and $9G^2=12H(I-3H^2)$.

The last condition requires that $3J+IH=0$.

Thus the required conditions are $I=0$ and $J=0$.

[$H=0$ makes $J=0$, $G=0$ and $I=0$ which are the conditions for four equal roots].

Ex. 1. Find the conditions for two distinct pairs of equal roots.

Ex. 2. Find the conditions that the biquadratic may have two positive and two negative roots.

Ex. 3. Discuss the nature of the roots of

$$x^4-9x^2+3x-2=0.$$

CHAPTER VI

Solution of Numerical Equations

§ 57. **Theorem.** *An equation with integral coefficients and unity for the coefficient of the first term, cannot have a commensurable root which is not an integer.*

If possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a root of the equation :

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0,$$

where the p's are all integers.

Then, we must have

$$\left(\frac{a}{b}\right)^n+p_1\left(\frac{a}{b}\right)^{n-1}+p_2\left(\frac{a}{b}\right)^{n-2}+\ldots+p_{n-1}\left(\frac{a}{b}\right)+p_n=0.$$

Multiplying by b^{n-1}, we get

$$-\frac{a^n}{b}=p_1a^{n-1}+p_2a^{n-2}b+p_3a^{n-3}b^2+\ldots+p_{n-1}ab^{n-2}+p_nb^{n-1} \qquad (1).$$

Since a and b have no common factor,

$$\frac{a^n}{b}$$

is a fraction in its lowest terms.

But, each term on the right hand side of (1) is an integer.

The above relation is thus absurd.

Hence $\frac{a}{b}$ is not a root of the given equation.

The real roots of the said equation are, therefore, either whole numbers or are incommensurable.

§ 58. Newton's Method of Divisors for obtaining the integral roots of an equation.

Suppose h is an integral root of the equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0, \qquad (i)$$

where the a's are all integers.

Let the quotient when $f(x)$ is divided by $(x-h)$, be

$$b_0x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \ldots\ldots + b_{n-1}.$$

Then

$$\sum_{r=0}^{n} a_r x^{n-r} \equiv (x-h) \sum_{k=0}^{n-1} b_k x^{n-k-1}.$$

Equating coefficients of like powers, we have

$$\begin{aligned}
&a_0 = b_0, \\
&a_1 = b_1 - b_0h, \quad \text{or} \quad a_1 - b_1 = -b_0h\,; \\
&a_2 = b_2 - b_1h, \quad \text{or} \quad a_2 - b_2 = -b_1h\,; \\
&\ldots\ldots\ldots\ldots \qquad\qquad \ldots\ldots\ldots\ldots \\
&a_r = b_r - b_{r-1}h, \quad \text{or} \quad a_r - b_r = -b_{r-1}h\,; \\
&\ldots\ldots\ldots\ldots \qquad\qquad \ldots\ldots\ldots\ldots \\
&a_{n-1} = b_{n-1} - b_{n-2}h, \text{ or } a_{n-1} - b_{n-1} = b_{n-2}h\,;
\end{aligned}$$

and $\quad a_n = -b_{n-1}h.$

It is evident that all the b's are integers. Also, from the last of these equations, it is clear that a_n is divisible by h, the quotient when a_n is divided by h being $-b_{n-1}$. The last but one equation shows that the sum of the second coefficient from the end in the given equation and the quotient obtained by dividing a_n by h, is again divisible by h, the quotient being $-b_{n-2}$ and so on. Continuing this process, the last quotient thus obtained will be $-b_0$ or which is the same thing $-a_0$. If we perform the process here indicated with all the divisors of a_n, positive as well as negative, then those of the divisors which satisfy the above conditions, giving integral quotients at all steps and a final quotient equal to $-a_0$, shall be the integral roots of $f(x)=0$. Those which give a fractional quotient at any stage are rejected outright.

When $a_0=1$, we know from the Theorem of the preceding section, that the integral roots so determined are the only commensurable roots of the given equation. To find all the commensurable roots of any equation with integral coefficients, we must, therefore, first transform it into another with integral

coefficients and unity for the coefficient of the first term. (*cf.* § 27, example 2).

§ 59. Newton's Method of Divisors is most conveniently employed by arranging the process as follows :—

$$\begin{array}{cccccccc} a_n+a_{n-1} & +a_{n-2} & +a_{n-3} & +\ldots & +a_2 & +a_1 & +a_0 \\ -b_{n-1} & -b_{n-2} & -b_{n-3} & -\ldots & -b_2 & -b_1 & -b_0 \\ \hline -hb_{n-1}-hb_{n-2} & -hb_{n-3} & -hb_{n-4} & -\ldots & -hb_1 & -hb_0 & -0 \end{array}$$

In the first line are written the successive coefficients $a_0, a_1, a_2 \ldots\ldots, a_n$ in the reverse order of their occurrence. The first figure in the third row is the same as the one in the first row. The figure $(-b_{n-1})$ is obtained by dividing the first figure in the third row by h. This is placed in the second row under a_{n-1} and gives on addition to it the second figure $(-hb_{n-2})$ in the third row. Any figure ($-b_r$, say) in the second row, is obtained by dividing the preceding figure in the third row, *viz.* $-hb_r$, by h. Any figure say $-hb_r$ in the third row is the sum of the two figures a_{r+1} and $-b_{r+1}$ above it. If h be a root, the last figure in the second line will be $-a_0$. This shall give zero as the last figure in the third row.

When we succeed in showing in this manner that an integer h is a root, the next operation with any divisor, may be performed not on the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots\ldots, a_0$ of the original polynomial but on those of the second line, for these are the coefficients (in the reverse order with all signs changed) in the quotient when the original polynomial is divided by $(x-h)$. The divisors $\pm$ 1 of a_n need not be included among the trial divisors. It is more convenient before applying the method of divisors, to find whether either of these is a root of the equation.

§ 60. If h be an integral root of the equation :

$$f(x)=\sum_{r=0}^{n} a_r x^{n-r}=0,$$

then, we have as before,

$$f(x)\equiv(x-h)\sum_{k=0}^{n-1} b_h x^{n-k-1}.$$

In this identity, if for x we put some positive or negative integer t, we get

$$f(t)=(t-h)\sum_{k=0}^{n-1} b_k t^{n-k-1},$$

showing that $f(t)$ is exactly divisible by $(t-h)$ or what is the same thing by $(h-t)$. In particular by putting $t=1$, we see that $f(1)$ is exactly divisible by $(h-1)$; and by putting $t=-1$, we find that $f(-1)$ is divisible by $(h+1)$.

Before testing any divisor h for a root, therefore, we subject it to the above conditions. Those of the divisors that fail to satisfy these conditions, are discarded at the very outset. Considering the superior and inferior limits of the real roots of $f(x)=0$, some more of the divisors may possibly be discarded.

Some of the integral roots of $f(x)=0$ may be equal. When an integer h has been found to be a root of $f(x)=0$ by Newton's method of divisors, it may be tried again with the coefficients of the reduced equation. It may be noticed that if h be a m-ple root of $f(x)=0$, a_n must be divisible by h^{m-1}.

Example 1. Find the integral roots of the equation:

$$f(x)\equiv x^4+x^3-2x^2+4x-24=0.$$

Here $f(0)=-24$, $f(1)=-20$ and $f(-1)=-30$.

The divisors (other than ± 1) of 24 are:

$$\pm(2, 3, 4, 6, 8, 12, 24). \qquad (i)$$

Adding unity to each of the divisors of $f(1)$, we get the set of numbers: $-19, -9, -4, -3, -1, 0, 2, 3, 5, 6, 11, 21$. *(ii)*

Subtracting unity from each of the divisors of $f(-1)$, we get another set of numbers *viz.*, $-31, -16, -11, -7, -6, -4, -3, -2, 0, 1, 2, 4, 5, 9, 14, 29$. *(iii)*

The only numbers common to the three sets are: -4, -3 and 2.

These are, therefore, the only possible integral roots of the given equation. Applying Newton's method of divisors:

$$
\begin{array}{r|r}
 & -24+\ 4-2+1+1 \\
 & +\ 6 \qquad\quad \\
\hline
-4 & -24+10 \qquad\quad \\
\hline
 & -24+\ 4-2+1+1 \\
 & +\ 8-4+2-1 \\
\hline
-3 & -24+12-6+3+0 \\
\hline
 & 8-4+2-1 \\
 & +4+0+1 \\
\hline
2 & 8+0+2+0
\end{array}
$$

we find that -3 and 2 are the only integral roots of $f(x)=0$.

The trial divisor -4 has to be rejected because it does not divide 10 exactly.

N.B.—The significance of the three sets of numbers employed in the solution of this example and the one following, will be readily seen in the light of the remarks made in the course of the above section.

Example 2. Find the integral roots of the equation :

$$f(x)\equiv x^4-2x^3-19x^2+68x-60=0.$$

We have $f(0)=-60, f(1)=-12$ and $f(-1)=-144$.

The three sets of numbers introduced above are :

(1) $-60,\ldots,-12, -10, -6, -5, -4, -3, -2, 2, 3, 4, 5, 6, 10, 12,\ldots, 60$;

(2) $-11, -5, -3, -2, -1, 0, 2, 3, 4, 5, 7, 13$;

(3) $-145,\ldots, -13, -10, -9, -7, -5, -4, -3, -2, 0, 1, 2, 3, 5, 7, 8, 11,\ldots, 143$.

The numbers common to the three sets are :

$$\pm 2, \pm 3 \text{ and } \pm 5.$$

Applying Newton's Method :

$$
\begin{array}{r|l}
 & -60+68-19-2+1 \\
 & \quad -30+19+0-1 \\
\hline
2 & -60+38+\ 0-2+0 \\
 & \quad -30+19+0-1 \\
 & \qquad -15+2+1 \\
\hline
2 & \quad -30+\ 4+2+0 \\
\hline
 & \qquad -15+2+1 \\
 & \qquad\quad -5-1 \\
\hline
3 & \qquad -15-3+0 \\
\hline
\end{array}
\qquad
\begin{array}{r|l}
 & -5-1 \\
 & \quad +1 \\
\hline
-5 & -5+0 \\
\hline
\end{array}
\qquad
\begin{array}{r|l}
 & -60+68-19-2+1 \\
 & \quad +30 \\
\hline
-2 & -60+98 \\
\hline
 & -15+2+1 \\
 & \quad +5 \\
\hline
-3 & -15+7 \\
\hline
\end{array}
$$

we find 2, 2, 3 and -5 to be the roots of the given equation.

The other divisors have to be rejected as shown above.

Example 3. Find the commensurable roots of the equation $f(x) \equiv 3x^4-23x^3+35x^2+31x-30=0$.

Using the method of § 46—47, we find that the real roots of $f(x)=0$, lie between -2 and 6.

Multiplying the roots of $f(x)=0$ by 3, we get the equation :

$$\phi(x) \equiv x^4-23x^3+105x^2+279x-810=0.$$

The roots of $\phi(x)=0$ must lie between -6 and 18.

Moreover, the integral roots of $\phi(x)=0$ are all its commensurable roots. Now the divisors of -810 which lie between -6 and 18 are $-6, -5, -3, -2, 2, 3, 5, 6, 9, 10, 15$ and 18.

Since $\phi(1)=-448$ and $\phi(-1)=-960$, of the above divisors

-5 is rejected because -448 is not divisible by 6 ;

-2 is rejected because -448 is not divisible by 3 ;

and 6 is rejected because -960 is not divisible by 7.

The divisors 10 and 18 are rejected for similar reasons.

We now apply Newton's method of divisors as follows :

	$-810+279+105-23+1$
	$+270-183+26-1$
-3	$-801+549-78+3+0$
	$270-183+26-1$
	$+135-24+1$
2	$270-48+2+0$
	$135-24+1$
	$+15-1$
9	$135-9+0$
	$15-1$
	$+1$
15	$15+0$

	$-810+279+105-23+1$
	$+135-69-6$
-6	$-810+414+36-29$
	$270-183+26-1$
	$-90+91-39$
-3	$270-273+117-40$
	$135-24+1$
	$+45+7$
3	$135+21+8$
	$135-24+1$
	$+27$
5	$135+3$

we find that -3, 2, 9 and 15 are the roots of $\phi(x)=0$.

Therefore, the roots of $f(x)=0$ are -1, $\frac{2}{3}$, 3 and 5.

Example 4. Solve the equation

$$f(x)\equiv 2x^6-15x^5+32x^4+23x^3-186x^2+260x-120=0,$$

by Newton's method of divisors.

Using the methods of § 46—47, the roots of the given equation are seen to lie between -3 and 3. Multiplying the roots of $f(x)=0$ by 2, we obtain the equation :

$$F(x)\equiv x^6-15x^5+64x^4+92x^3-1488x^2+1460x-3840=0.$$

The real roots of $F(x)=0$ all lie between -6 and 6,

Of the divisors of -3840, we have to test only such as lie between -6 and 6. These are $\pm(2, 3, 4, 5, 6)$.

Since $F(-1)=-9580$ is not divisible by 3, -3, 6 and 7, the divisors 2, -4, 5 and 6 are rejected.

Again, since $F(1)=-1026$ is not divisible by -4 and -7, the divisors -5 and -6 are also rejected.

Considering the remaining divisors 3, 4, -2 and -5 in order

$$
\begin{array}{r|r}
 & -3840+4160-1488+92+64-15+1 \\
 & -1280+960-176-28+12-1 \\
\hline
3 & -3840+2880-528-84+36-3+0 \\
\hline
 & -1280+960-176-28+12-1 \\
 & -320+160-4-8+1 \\
\hline
4 & -1280+640-16-32+4+0 \\
\hline
 & -320+160-4-8+1 \\
 & -80+20+4-1 \\
\hline
4 & -320+80+16-4+0 \\
\hline
 & -80+20+4-1 \\
 & -20+0+1 \\
\hline
4 & -80+0+4+0 \\
\hline
\end{array}
$$

we find that 3 is a root of $F(x)=0$, while 4 is found to be a triple root of the same equation. The remaining two roots occur in the quadratic $x^2-20=0$, and are $-2\sqrt{5}$ and $2\sqrt{5}$.

Therefore, the roots of the equation $f(x)=0$ are
$\frac{3}{2}$, 2, 2, 2, $\pm\sqrt{5}$.

EXERCISES XXXI

Find the commensurable roots (if any) of the following equations :—

1. $x^5-13x^4+67x^3-171x^2+216x-108=0$.
   Ans. 2, 2, 3, 3, 3.

2. $x^4-2x^3-13x^2+38x-24=0$.
   Ans. 1, 2, 3, -4.

3. $x^5-29x^4-31x^3+31x^2-32x+60=0$.
   Ans. 1, -2, 30.

4. $x^5-23x^4+160x^3-281x^2-257x-440=0$.
   Ans. 5, 8, 11.

5. $2x^3-31x^2+112x+64=0$.
   Ans. $-1/2$, 8, 8.

6. $x^4+12x^3+32x^2-24x+24=0$.
   Ans. None.

7. $x^5-x^4-12x^3+8x^2+28x+12=0.$

Ans. -3.

8. $2x^5-5x^4+2x^3+9x^2+4x-12=0.$

Ans. 1.

§ 61. Horner's method for finding the commensurable and incommensurable Roots of an Equation.

The main principle involved in this method is the successive diminution of the roots of the given equation by suitable numbers. The root is evolved figure by figure : first the integral part (if any), and then the decimal part, till the root either terminates or is calculated to the required number of decimal places. Each new figure in the root is found by trial. If, for example, the required root be 24·302, then the roots shall be successively diminished by 20, 4, ·3 and ·002. The transformations are all affected by Horner's Process. One great advantage of this method is that the successive transformations are all exhibited in a compact arithmetical form.

Example. Find by Horner's Method the positive root of the equation $16x^3-20x^2-50x-375=0$.

The equation cannot have more than one positive root, there being only one change of sign in the polynomial

$$f(x)\equiv 16x^3-20x^2-50x-375.$$

By trial, we find that this root lies between 3 and 4.

Diminishing the roots of $f(x)=0$ by 3, we get the equation

$$f_1(x)\equiv 16x^3+124x^2+262x-273=0,$$

the positive root of which must lie between 0 and 1.

Multiplying the roots of $f_1(x)=0$ by 10, we get

$$f_2(x)\equiv 16x^3+1240x^2+26200x-273000=0$$

whose positive root must lie between 0 and 10.

We see by trial that the equation $f_2(x)=0$ has a root between 7 and 8. The next figure in the root of $f(x)=0$ is therefore, 7.

Diminishing the roots of $f_2(x)=0$ by 7, we have

$$f_3(x)\equiv 16x^3+1576x^2+45912x-23352=0,$$

which must have a root between 0 and 1.

Multiplying the roots of $f_3(x)=0$ by 10, we obtain

$$f_4(x) \equiv 16x^3+15760x^2+4591200x-23352000=0,$$

the positive root of which must lie between 0 and 10.

By trial again, we find that 5 is a root of $f_4(x)=0$.

Thus, we see that the positive root of $f(x)=0$ is 3·75.

The process is exhibited as follows :

16	−20	−50	−375	(3·75
	48	84	102	
	28	34	−273000	
	48	228	249648	
	76	26200	−23352000	
	48	9464	23352000	
	1240	35664	0	
	112	10248		
	1352	4591200		
	112	79200		
	1464	4670400		
	112			
	15760			
	80			
	15840			

EXERCISES XXXII

Find the positive roots of the following equations :

1. $20x^3-121x^2-121x-141=0$. Ans. 7·05.
2. $2x^3-85x^2-85x-87=0$. Ans. 43·5
3. $4x^3-13x^2-31x-275=0$. Ans. 6·25

§ 62. **Newtonis method of approximation.** It will be of importance to notice that after two or three transformations and sometimes, even earlier, the figures in the root are obtained by dividing the first coefficient from the end by the second coefficient from the end. This hint of practical value, is based on the following theorem :—

If a root of the equation $f(x)=0$ differs from α by a small quantity h, then h is approximately equal to

$$-\frac{f(\alpha)}{f'(\alpha)}.$$

We have

$$f(\alpha+h)\equiv f(\alpha)+hf'(\alpha)+\frac{h^2}{2\,!}f''(\alpha)+\ldots+\frac{h^r}{r\,!}f^{(r)}(\alpha)+\ldots=0$$

because $(\alpha+h)$ is a root of the equation $f(x)=0$.

Since h is a small quantity, neglecting its square and higher powers, we have as an approximation

$$f(\alpha)+hf'(\alpha)=0, \text{ so that } h=-\frac{f(\alpha)}{f'(\alpha)}.$$

When the roots of the equation $f(x)=0$ have been diminished by α, the last coefficient in the transformed equation is $f(\alpha)$ and the last coefficient but one is $f'(\alpha)$. Hence it is that the figures in the root, after a few transformations, begin to be suggested by mere division of the second of these two coefficients by the first *i.e.* by the quotient of $-f(a)$ by $f'(\alpha)$.

If h is positive, as it is in the application of Horner's Method described above, it is essential that when the figures in the root begin to suggest themselves (as stated above), the last two coefficients in the transformed equation have opposite signs.

Example. The cubic $x^3-3x+1=0$ has a root between 1 and 2. Calculate it to five places of decimals.

The first figure in the root is 1.

Diminishing the roots of the given equation by 1 and multiplying by 10 the roots of the equation so obtained, we get

$$x^3+30x^2-1000=0. \qquad \ldots \qquad (i)$$

This has a root betweent 5 and 6, therefore 5 is the second figure in the root. The third and the succeeding figures in the root are suggested by a division of the absolute term in the transformed equation by the coefficient of the preceding term. The third figure 3 is, thus, obtained by dividing 125000 by 37500; the fourth figure 2 is similarly obtained by dividing 8423000 by 4022700 ; the fifth figure is a zero because 359232000 is less than 404107200 ; and the sixth figure is 8 because $\frac{359232000000}{40410720000}$ lies between 8 and 9.

The required root is, therefore, 1·53208......

The process is exhibited as follows :—

1	$+0$	-3	$+1$	(1·53208...
	1	1	-2	
	1	-2	-1000	
	1	2	875	
	2	000	-125000	
	1	175	116577	
	30	175	-8423000	
	5	200	8063768	
	35	37500	-359232000000	
	5	1359		
	40	38859		
	5	1368		
	450	4022700		
	3	9184		
	453	4031884		
	3	9188		
	456	40410720000		
	3			
	4590			
	2			
	4592			
	2			
	4594			
	2			
	459600			

EXERCISES XXXIII

1. Calculate to five decimal places the two roots of the equation $x^3-3x+1=0$ which lie between 0 and 1; and -1 and -2. **Ans.** $\cdot 34729\ldots$, $-1\cdot 87938\ldots$

2. Calculate to four decimal places the positive roots of the equation $x^4+4x^3-4x^2-11x+4=0$.

Ans. $1\cdot 6369\ldots$, $\cdot 3373\ldots$

3. The equation $x^4-3x^2+75x-10000=0$ has a root between 9 and 10. Calculate it to three places of decimals.

Ans. $9\cdot 886$.

4. Find to four decimal places the positive root of the equation $x^3+x^2+x-100=0$.

Ans. $4\cdot 2644\ldots\ldots$

5. Calculate to five decimal places the positive root of the equation $x^3-2x-5=0$.

Ans. $2\cdot 09455\ldots\ldots$

§ 63. Contraction of Horner's Method.

In the ordinary process of contracted division, when the given figures in the dividend are exhausted, instead of appending ciphers to it, we cut off figures successively from the right of the divisor, so that divisor itself is exhausted. In Horner's contracted method the principle is the same. When the contracted process begins, instead of appending ciphers to the coefficients of the tansformed equation, we cut off one figure from the right of the last coefficient but one (called the trial divisor), two figures from the right of the last coefficient but two and so on. The stage at which the contracted process should begin depends upon the number of decimal places required in the answer; for after the

contraction begins we shall get, in addition to the figures already found, as many more figures as there are figures in the trial divisor less one.

In multiplying by the corresponding figure of the root, the figures cut off should be multiplied mentally and account taken of the number to be carried, just as in contracted division.

The following examples will make the method clear.

Example 1. The equation $x^4+12x+7=0$ has a root between -1 and 0. Calculate it correctly to seven decimal places.

Changing the signs of the roots of the given equation, we get the equation

$$x^4-12x+7=0. \qquad \ldots \qquad \text{(i)}$$

This must have a root between 0 and 1 corresponding to the proposed root of the given equation. We start by multiplying the roots of (i) by 10, and calculate the root under consideration as follows :—

+0	+0	−12000	+70000 (·59368583...
5	25	125	−59375
5	25	−11875	106250000
5	50	375	−102132639
10	75	−11500000	41173610000
5	75	151929	−33516590799
15	15000	−11348071	7657019201
5	1881	169587	−6698773074
200	16881	−11178484000	958246127
9	1962	6287067	−893054808
209	18843	−11172196933	65191319
9	2043	6308361	−55815025
218	2088600	−11165888572	9376294
9	7089	126678	−8930395
227	2095689	−1116462179	445899
9	7098	126762	−334890
2360	2102787	−1116335417	−111009
3	7107	1691	
2363	2109894	−111631851	
3	14	1691	
2366	21113	−111630160	
3	14	11	
2369	21127	−11163005	
3	14	11	
2372	21141	−11162994	
		−111630	

Hence the required root is −·5936858......

Example 2. Find correctly to ten decimal places the cube root of 25.

We have to calculate the real root of the equation

$$f(x) \equiv x^3 - 25 = 0.$$

As a first approximation, we have 3 as the real root of $f(x) = 0$.

If the correct value of the root be $(3-h)$, then h is approximately equal to

$$\frac{f(3)}{f'(3)} = \frac{27-25}{3.9} = \frac{2}{27} = \cdot 047\ldots\ldots$$

Thus 2·926 is a closer approximation to the root.

If the correct value of the root be $(2\cdot 926 - h_1)$, then h_1 is nearly equal to

$$\frac{f(2\cdot 926)}{f'(2\cdot 926)} = \frac{(2\cdot 926)^3 - 25}{3.(2\cdot 926)^2} = \frac{\cdot 050949}{25\cdot 6844} = \cdot 00198.$$

Thus $\sqrt[3]{25}$ is nearly equal to 2·92402.

Using Horner's Method now

1	+0	+0	−25 (2·92401773821...
	2	4	8
	2	4	−17000
	2	8	16389
	4	1200	−611000
	2	621	508088
	60	1821	−102912000
	9	702	102457024
	69	252300	−454976000000
	9	1744	256494157201
	78	254044	−198481842799
	9	1748	179546953908
	870	25579200	−18934888891
	2	35056	17954742673
	872	25614256	−980146218
	2	35072	769489164
	874	256493280000	−210657054
	2	877201	205197112
	8760	256494157201	−5459942
	4	877202	5129928
	8764	256495034403	−330014
	4	61404	256496
	8768	25649564844	−73518
	4	61404	
	877200	25649626248	
	1	614	
	877201	2564963239	
	1	614	
	877202	2564963853	
	1	3	
	877203	256496388	
		3	
		256496391	

we get $\sqrt[3]{25} = 2{\cdot}9240177382\ldots\ldots$

EXERCISES XXXIV

1. Calculate to 8 decimal places the value of

(*i*) $\sqrt{37}$, (*ii*) $\sqrt[3]{998}$.

Ans. (*i*) 6·08276253. (*ii*) 9·99332888.

2. Find the cube root of 43·651389761. **Ans.** 3·521.

3. Calculate to eight decimal places the two roots of

$$x^3-7x+7=0$$

which lie between 1 and 2. **Ans.** 1·35689584, 1·69202147.

4. Calculate to ten decimal places, the root of

$$x^3-4x^2+6x-20000=0$$

which lies between 20 and 30. **Ans.** 28·4678477449.

§ **64. Lagrange's Method.** This method too like that of Horner, consists in the successive transformation of the given equation. Suppose the equation $f(x)=0$ has a positive root between the two consecutive integers a and $a+1$. The roots of $f(x)=0$ can then be diminished by a. The transformed equation has a root between 0 and 1. Suppose the equation whose roots are the reciprocals of the roots of this transformed equation, has a root between the two consecutive integers b and $b+1$, where $b>1$, we can then, diminish the roots by b and reciprocate the roots of the transformed equation. This process of diminution and reciprocation of roots is carried on and we get the root of the given equation in the form : $a+\dfrac{1}{b+}\,\dfrac{1}{c+}\,\dfrac{1}{d+}\cdots\cdots$ which is a continued fraction.

Example. Find by Lagrange's Method, the positive root of the equation $f(x)\equiv x^2-6x-13=0$.

The equation has only one positive root. By trial we find that it lies between 3 and 4. Diminishing the roots of $f(x)=0$ by 3, we get the equation :

$$f_1(x)\equiv x^3+9x^2+21x-4=0.$$

The equation whose roots are the reciprocals of those of $f_1(x)=0$ is $f_2(x)\equiv-4x^3+21x^2+9x+1=0$.

This has a root between 5 and 6.

Diminishing the roots of $f_2(x)=0$ by 5 and reciprocating the roots of the equation so obtained, we get

$$f_3(x)\equiv 71x^3-81x^2-39x-4=0.$$

This has a root between 1 and 2.

The next set of transformations brings the equation :

$$f_4(x)\equiv -53x^3+12x^2+132x+71=0,$$

which has also, a root between 1 and 2.

We next obtain the equation

$$f_5(x)\equiv 162x^3-3x^2-147x-53=0,$$

which too has a root between 1 and 2.

Thus the required root is

$$3+\frac{1}{5+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\cdots\cdots$$

The whole process may be exhibited as follows :

	1	+0	−6	−13	(3
		3	9	9	
		3	3	−4	
		3	18		
		6	21		
		3	−20		
		9	1		
		5	−20		
5)	1	14	−19		
	70	−95	−20		
	71	−81	−39	− 4	(1
		71	−10	−49	
		−10	−49	−53	
		71	61		
		61	12		
		71	−53		
		132	−41		
		−41	−53		
1)	71	91	−94		
	91	−94	−53		
	162	−3	−147	−53	(1

EXERCISES XXXV

1. Find by Langrange's method the positive roots of

(i) $x^3-2x-5=0.$ Ans. $2+\frac{1}{10+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\cdots\cdots$

(ii) $x^3+2x^2-23x-70=0.$

Ans. $5+\frac{1}{7+}\frac{1}{2+}\frac{1}{3+}\frac{1}{9+}\cdots\cdots$

2. Find by Langrange's method the value of :

(i) $\sqrt[3]{25}$, Ans. $2+\frac{1}{1+}\frac{1}{12+}\frac{1}{6+}\frac{1}{4+}\cdots\cdots$

(ii) $\sqrt{37}$. Ans. $6+\frac{1}{12+}\frac{1}{12+}\frac{1}{12+}\frac{1}{12+}\cdots\cdots$

CHAPTER VII

The Cubic, the Biquadratic and the Binomial.

§ 65. Reduction of the General Cubic to the Standard Form.

The General Cubic equation is of the form :

$$\phi_3(x) \equiv p_0x^3 + 3p_1x^2 + 3p_2x + p_3 = 0.$$

Diminishing the roots of this equation by h, we get

$$\phi_3(x+h) \equiv \phi_0(h)x^3 + 3\phi_1(h)x^2 + 3\phi_2(h)x + \phi_3(h) = 0\;;$$

i.e., $p_0x^3 + 3(p_0h + p_1)x^2 + 3(p_0h^2 + 2p_1h + p_2)x$
$+ (p_0h^3 + 3p_1h^2 + 3p_2h + p_3) = 0$

The second term will disappear if $h = -p_1/p_0$.

The equation, thus, takes the form :

$$\phi_3\left(x - \frac{p_1}{p_0}\right) \equiv p_0x^3 + 3\left[\frac{p_0p_2 - p_1^2}{p_0}\right]x + \left[\frac{p_3p_0^2 - 3p_0p_1p_2 + 2p_1^3}{p_0^2}\right] = 0$$

Multiplying the roots by p_0, we get

$$p_0^2\phi_3[p_0(x - p_1)] \equiv x^3 + 3(p_0p_2 - p_1^2)x + (p_3p_0^2 - 3p_0p_1p_2 + 2p_1^3) = 0.$$

Denoting $p_0p_2 - p_1^2$ or $\begin{vmatrix} p_0 & p_1 \\ p_1 & p_2 \end{vmatrix}$ by H, and

$p_3p_0^2 - 3p_0p_1p_2 + 2p_1^3$ by G, the equation finally becomes

$$x^3 + 3\mathrm{H}x + \mathrm{G} = 0.$$

This is called the standard form of the cubic.

If the roots of the general cubic be denoted by α, β, γ then the roots of the standard cubic are

$$p_0\alpha + p_1,\ p_0\beta + p_1,\ p_0\gamma + p_1.$$

Thus the solution of the general cubic depends upon that of the standard cubic.

Ex. 1. If α, β, γ be the roots of the general cubic, show that those of the standard cubic are

$$\frac{p_0}{3}(2\alpha-\beta-\gamma),\ \frac{p_0}{3}(2\beta-\gamma-\alpha),\ \frac{p_0}{3}(2\gamma-\alpha-\beta).$$

Ex. 2. Prove that $\Sigma(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)=\dfrac{27H}{p_0^{\,2}}$,

and
$$(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)=-\frac{27G}{p_0^{\,3}}.$$

Example. Reduce the cubic

$2x^3-3x^2+10x-4=0$, to the standard form.

Multiplying by 2, the roots of the given equation, we get

$$x^3-3x^2+20x-16=0. \qquad (i)$$

Diminishing by 1, the roots of (i),

$$\begin{array}{rrrr}
1 & -3 & +20 & -16 \\
 & 1 & -2 & +18 \\
\hline
 & -2 & +18 & \multicolumn{1}{|r}{+2} \\
 & 1 & -1 & \\
\hline
 & -1 & \multicolumn{1}{|r}{+17} & \\
 & 1 & & \\
\hline
 & 0 & &
\end{array}$$

we obtain $x^3+17x+2=0$.

This is the standard form of the given cubic.

EXERCISES XXXVI

Reduce the following cubics to the standard form :

1. $5x^3-3x^2+9x-2=0$.
2. $7x^3-12x^2-6x-5=0$.
3. $4x^3-2x^2-3x+2=0$.
4. $15x^3-3x^2+2=0$.
5. $x^3-x^2-x+1=0$.

§ 66. Criterion of the Nature of the Roots of the Cubic.

The equation of 'squared differences' of the cubic

$$f(x)=x^3+3Hx+G=0, \qquad \text{(i)}$$

is $F(x)=x^3+18Hx^2+81H^2x+27(G^2+4H^3)=0.$ (ii)

Since imaginary roots occur in pairs, at least one of the three roots of (i) must be real. Let α, β, γ denote the roots of (i).

(a) If $G^2+4H^3=0$, one root, at least, of $F(x)=0$ is zero. Therefore, at least two roots of the given cubic are equal.

If $H\neq 0$ and $G^2+4H^3=0$ only two roots of the given cubic are equal.

If $H=0$ and $G=0$, all the roots of $F(x)=0$ are equal to zero and, therefore, the given cubic has its three roots equal.

[The expression G^2+4H^3 is called the 'discriminant' of the cubic and its vanishing provides the condition for equal roots.]

(b) If G^2+4H^3 is negative, H is necessarily negative. Hence there is no change of sign in the expression

$$-F(-x)\equiv x^3-18Hx^2+81H^2x-27(G^2+4H^3)$$

and the equation of squared differences, $F(x)=0$ can have no negative root.

$$\beta-\gamma,\ \gamma-\alpha,\ \alpha-\beta \text{ are, thus, all real numbers.}$$

All the roots of $f(x)=0$ are, therefore, real and distinct.

(c) If G^2+4H^3 is positive, the product

$$(\alpha-\beta)^2\ (\beta-\gamma)^2\ (\gamma-\alpha)^2$$

of the roots of $F(x)=0$ is negative. Therefore, two roots of $f(x)=0$ are imaginary, for if α, β, γ were all real, $(\alpha-\beta)^2\ (\beta-\gamma)^2\ (\gamma-\alpha)^2$ could not be negative. Moreover $\alpha\beta\gamma=-G$, therefore, the sign of the real root, in this case, is opposite to that of G.

Ex. 1. Show that the roots of the equation of squared differences of the cubic are either all positive or one negative and two imaginary.

Ex. 2. Show that for the general cubic
$$\phi_3(x) \equiv p_0x^3+3p_1x^2+3p_2x+p_3=0,$$
the conditions $G=0$, $H=0$ reduce to $\dfrac{p_0}{p_1}=\dfrac{p_1}{p_2}=\dfrac{p_2}{p_3}$ and make $\phi_3(x)$ a perfect cube.

Ex. 3. Determine the nature of the roots of the cubic
$$x^3-7x+6=0.$$

Solution. Here $H=-7/3$, $G=6$.

Therefore, $G^2+4H^3=36-\frac{1372}{27}=-\frac{400}{27}$.

Hence, the roots of the given cubic are all real.

EXERCISES XXXVII

Determine the nature of the roots of the following equations :

1. $x^3-3x^2-6x+5=0$. **Ans.** All the roots are real.
2. $x^3+6x^2+9x+4=0$. **Ans.** Two roots are equal.
3. $x^3+2x+1=0$. **Ans.** The only real root is negative.
4. $x^3+x^2-5x+3=0$. **Ans.** Two roots are equal.

§ 67. Cardan's Solution of the Cubic.

Consider the standard cubic $x^3+3Hx+G=0$. **(i)**

Assume that $x=p^{1/3}+q^{1/3}$, then taking the cube of both sides, we have
$$x^3=p+q+3p^{1/3}q^{1/3}(p^{1/3}+q^{1/3}),$$
or
$$x^3-3p^{1/3}q^{1/3}x-(p+q)=0. \quad \textbf{(ii)}$$

If equations (i) and (ii) be identical, we have
$$p+q=-G \text{ and } p^{1/3}q^{1/3}=-H \quad \text{i.e. } pq=-H^3.$$

Therefore, $(p-q)^2=(p+q)^2-4pq=G^2+4H^3$.

Hence $$p=\tfrac{1}{2}[-G+\sqrt{G^2+4H^3}]. \quad \textbf{(iii)}$$

Also $$q^{1/3}=-\frac{H}{p^{1/3}}.$$

If G^2+4H^3 be not negative, then corresponding to the three cube-roots of unity, (*iii*) gives three values for $p^{1/3}$ *viz.*

$$\sqrt[3]{p},\ w\sqrt[3]{p} \text{ and } w^2\sqrt[3]{p}.$$

Hence, the three roots of the standard cubic are

$$\sqrt[3]{p}-\frac{H}{\sqrt[3]{p}},\ w\sqrt[3]{p}-\frac{H}{w\sqrt[3]{p}} \text{ and } w^2\sqrt[3]{p}-\frac{H}{w^2\sqrt[3]{p}}$$

i.e. $$\sqrt[3]{p}-\frac{H}{\sqrt[3]{p}},\ w\sqrt[3]{p}-\frac{w^2H}{\sqrt[3]{p}} \text{ and } w^2\sqrt[3]{p}-w\frac{H}{\sqrt[3]{p}}.$$

It may be noted that the algebraic solution of the cubic given above is of little use in solving numerical equations.

It is only when G^2+4H^3 is positive or zero, that the value of p given by (*iii*), is real and $\sqrt[3]{p}$ has an arithmetical meaning. This is the case, when the cubic has either two imaginary or two equal roots. But when G^2+4H^3 is negative, the case in which the roots of the cubic are all real and distinct, p is a complex number and $\sqrt[3]{p}$ has, then, no arithmetical meaning. This is called the 'irreducible case of Cardan's solution'. We consider this case in the next section.

§ 68. The Irreducible case of Cardan's Solution.

To find the value of $p^{1/3}$ when G^2+4H^3 is negative, we make use of De Moivre's Theorem in Trigonometry.

Let $-\dfrac{G}{2}=a$ and $\dfrac{\sqrt{-G^2-4H^3}}{2}=b,$

so that $$p=a+ib.$$

Put $$a=r\cos\theta \text{ and } b=r\sin\theta.$$

Then $$p=r(\cos\theta+i\sin\theta),$$

where $$r=\sqrt{a^2+b^2}=\sqrt{-H^3}$$

and θ is given by the equations

$$\sin\theta=\frac{b}{r}=\sqrt{1+\frac{G^2}{4H^3}}.$$

and
$$\cos\theta=\frac{a}{r}=-\frac{G}{2\sqrt{-H^3}}.$$

The three values of $p^{1/3}$, now, are

$$\sqrt[3]{r}\left(\cos\frac{\theta+2n\pi}{3}+i\sin\frac{\theta+2n\pi}{3}\right),\ n=0,1,2:$$

or
$$\sqrt{-H}\,e^{i\left(\frac{\theta+2n\pi}{3}\right)},\ n=0,1,2.$$

The roots of the standard cubic, obtained by substituting in succession each of these values of $p^{1/3}$ in $\left(p^{1/3}-\frac{H}{p^{1/3}}\right)$ are

$2\sqrt{-H}\cos\theta$, $2\sqrt{-H}\cos\frac{\theta+2\pi}{3}$ and $2\sqrt{-H}\cos\frac{\theta+4\pi}{3}$.

This completes the solution of the cubic $x^3+3Hx+G=0$.

Example 1. Solve the cubic $x^3-18x-35=0$.

Assume
$$x=p^{1/3}+q^{1/3},$$

then
$$x^3-3p^{1/3}q^{1/3}x-(p+q)=0.$$

Comparing coefficients, we have

$$p+q=35 \text{ and } p^{1/3}q^{1/3}=6 \text{ i.e. } pq=216.$$

p and q are, therefore, the roots of the quadratic
$$x^2-35x+216=0.$$

Hence $p=27$ or 8;

and $p^{1/3}=3, 3w$ or $3w^2$; or $2, 2w, 2w^2$.

Moreover $q^{1/3}=\frac{6}{p^{1/3}}=2, 2w^2, 2w$; or $3, 3w^2, 3w$.

In either case

$$5,\ 3w+2w^2,\ 3w^2+2w,$$

are the three roots of the given cubic.

[It would be seen that it is immaterial which value of p we take. The value of $q^{1/3}$ must, however, be so chosen that the second relation *e.g.* $p^{1/3}q^{1/3}=6$, in this case, is satisfied].

Example 2. Solve the cubic $x^3-15x^3-33x+847=0$.

To remove the second term, we diminish the roots by 5.

$$\begin{array}{llll}
1 & -15 & -33 & +847 \\
 & 5 & -50 & -415 \\
\hline
 & -10 & -83 & +432 \\
 & 5 & -25 & \\
\hline
 & -5 & -108 & \\
 & 5 & & \\
\hline
 & 0 & &
\end{array}$$

We thus get the equation

$$y^3-108y+432=0 \text{ where } y=x-5. \qquad (i)$$

To solve (i), we assume $y=p^{1/3}+q^{1/3}$. Then

$$y^3-3p^{1/3}q^{1/3}y-(p+q)=0.$$

Comparing coefficients, we get

$$p+q=-432,\ p^{1/3}q^{1/3}=36 \textit{ i.e. } pq=36^3.$$

Hence $p=-216$, so that $p^{1/3}=-6,\ -6w,\ -6w^2$.

Now $$q^{1/3}=\frac{36}{p^{\frac{1}{3}}}=-6,\ -6w^2,\ -6w.$$

Therefore $y=-12,\ -6(w+w^2),\ -6(w+w^2)$

$$\textit{i.e. } y=-12,\ 6,\ 6.$$

Hence $$x=-7,\ 11,\ 11, \text{ from } (i).$$

Example 3. Solve the cubic $x^3-6x-4=0$.

Let $$x=p^{1/3}+q^{1/3}.$$

then $$x^3-3p^{1/3}\,q^{1/3}\,x-(p+q)=0.$$

Comparing coefficients, we obtain

$$p+q=4 \text{ and } p^{1/3}\,q^{1/3}=2 \qquad i.e.\ pq=6.$$

Hence $p=2+\sqrt{4-8}=2(1+i)$;

$$=2\sqrt{2}[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}].$$

Therefore, $$p^{1/3}=\sqrt{2}[\cos\frac{2n\pi+\frac{\pi}{4}}{3}+i\sin\frac{2n\pi+\frac{\pi}{4}}{3}],$$

$$n=0,\ 1,\ 2.$$

Also $$q^{1/3}=\frac{2}{p^{\frac{1}{3}}}=\sqrt{2}[\cos\frac{2n\pi+\frac{\pi}{4}}{3}-i\sin\frac{2n\pi+\frac{\pi}{4}}{3}].$$

$$n=0,\ 1,\ 2.$$

Thus, the three roots of the given cubic are

$$2\sqrt{2}\cos\frac{2n\pi+\frac{\pi}{4}}{3},\ n=0,\ 1,\ 2\ ;$$

i.e. $$2\sqrt{2}\cos\frac{\pi}{12},\ 2\sqrt{2}\cos\frac{3\pi}{4},\ 2\sqrt{2}\cos\frac{17\pi}{12}\ ;$$

or $$\sqrt{3}-1,\ -2,\ -\sqrt{3}+1$$

Example 4. Find the relation between H and G so that the cubic $x^3+3Hx+G=0$, may be put in the form

$$x^4=(x^2+ax+b)^2.$$

Hence, solve the equation $8x^3-36x+27=0$.

If the two equations $x^3+3Hx+G=0$ and $x^4=(x^2+ax+b)^2$ are identical, we must have on comparing eoefficients

$$\frac{2a}{1}=\frac{a^2+2b}{0}=\frac{2ab}{3H}=\frac{b^2}{G}$$

i.e. $a^2+2b=0$, $3H=b$, $G=\dfrac{b^2}{2a}$.

Eliminating a and b between these equations, we obtain the required relation viz. $8G^2+27H^3=0$.

This relation is satisfied by the coefficients of the equation

$$8x^3-36x+27=0$$

The equation can, therefore, be put in the form :

$$x^4=(x^2+ax+b)^2.$$

Here, we have $H=-\frac{3}{2}$ and $G=\frac{27}{8}$.

Hence $b=3H=-\frac{9}{2}$ and $a=\dfrac{b^2}{2G}=3$.

The given equation, therefore, takes the form

$$x^4-(x^2+3x-\tfrac{9}{2})^2=0,$$

i.e.

$$(3x-\tfrac{9}{2})\,(2x^2+3x-\tfrac{9}{2})=0.$$

Hence $\frac{3}{2}$, $\dfrac{-3\pm3\sqrt{5}}{4}$ are the roots of the given cubic.

EXERCISES XXXVIII

Solve :—

1. $25x^3-9x^2+1=0$. **Ans.** $-{}^1/_4$, $\dfrac{2\pm i\sqrt{3}}{7}$.
2. $2x^3+3x^2+3x+1=0$. **Ans.** $-{}^1/_2$, $\dfrac{-1\pm i\sqrt{3}}{2}$.
3. $x^3+72x-1720=0$. **Ans.** $10, -5\pm7i\sqrt{3}$.
4. $x^3-11x^2+38x-40=0$. **Ans.** $2, 4, 5$.
5. $x^3+6x^2+9x+4=0$. **Ans.** $-1, -1, -4$.
6. $x^3-3x+1=0$. **Ans.** $2\cos\dfrac{2\pi}{9}$, $\cos\dfrac{8\pi}{9}$, $2\cos\dfrac{14\pi}{9}$

§ 69. Other Methods of solving the Cubic.

*I. Consider the cubic $x^3+3Hx+G=0$.

* American Mathematical Monthly, 1925.

The equation can be written in the form :

$$\frac{x^3}{8}+\frac{3x^3}{8}+\frac{3}{2}Hx=-\frac{G}{2},$$

or

$$\frac{x^3}{8}+\frac{3}{2}x\left(\frac{x^2}{4}+H\right)=-\frac{G}{2}.$$

Writing y^2 for $\frac{x^2}{4}+H$, this becomes

$$\frac{x^3}{8}+\frac{3}{2}xy^2=-\frac{G}{2}. \qquad (i)$$

Now let $\frac{3}{4}x^2y+y^3=R$ (ii)

where R remains to be determined.

From (*i*) and (*ii*), we get by addition and subtraction

$$\left(\frac{x}{2}+y\right)^3=-\frac{G}{2}+R, \qquad (iii)$$

and

$$\left(\frac{x}{2}-y\right)^3=-\frac{G}{2}-R. \qquad (iv)$$

Multiplying (*iii*) and (*iv*), and using the relation

$$\frac{x^2}{4}+H=y^2,$$

we obtain

$$-H^3=\frac{G^2}{4}-R^2. \qquad (v)$$

Hence $R=\pm\frac{1}{2}\sqrt{G^2+4H^3}.$ (vi)

From (*iii*), $\frac{x}{2}+y=\left(-\frac{G}{2}+R\right)^{1/3},$ (vii)

and from (*iv*), $\frac{x}{2}-y=\left(-\frac{G}{2}-R\right)^{1/3}.$ (viii)

Since $\frac{x^2}{4}-y^2=-H$, (*viii*) can be written more usefully in the form :

$$\frac{x}{2}-y=-\frac{H}{\left(-\frac{G}{2}+R\right)^{\frac{1}{3}}} \qquad (ix)$$

Adding (*vii*) and (*ix*), we finally get

$$x=\left(-\frac{G}{2}+R\right)^{1/3}-\frac{H}{\left(-\frac{G}{2}+R\right)^{1/3}}.$$

This gives three values of x corresponding to the three cube roots of $\left(-\frac{G}{2}+R\right)$. From (*vii*) and (*viii*), it is evident that any of the two values of R given by (*vi*), will do.

Ex. 1. Solve the cubic $x^3-9x^2+20x-12=0$.

Ans. 1, 2, 6.

Ex. 2. Find a cube root of $9+25i\sqrt{2}$.

Solution. Let $\frac{x}{2}+y=(9+25i\sqrt{2})^{1/3}$

and $\frac{x}{2}-y=(9-25i\sqrt{2})^{1/3}$;

so that $-G=18$, and $-H=\frac{x^2}{4}-y^2=[(9)^2-(25i\sqrt{2})^2]^{1/3}=11$.

Hence x is a root of the equation

$$x^3-33x-18=0.$$

One value of x is found to be 6 by trial.

Substituting this value of x in the relation

$$\frac{x^2}{4}-y^2=11,$$

we get $$y=i\sqrt{2}.$$

Hence $3+i\sqrt{2}$ is a cube root of $9+25i\sqrt{2}$.

II. Expressing the Cubic expression $x^3+3Hx+G$ as the difference of two cubes.

Let $x^3+3Hx+G\equiv L(x-\alpha)^3-M(x-\beta)^3$.

Then equating coefficients of like powers of x, we obtain

$$L-M=1, \tag{i}$$
$$\alpha L-\beta M=0, \tag{ii}$$
$$\alpha^2L-\beta^2M=H. \tag{iii}$$
$$\alpha^3L-\beta^3M=-G. \tag{iv}$$

From (*i*) and (*ii*), we get

$$L=\frac{\beta}{\beta-\alpha},\ M=\frac{\alpha}{\beta-\alpha}. \tag{v}$$

Substituting in (*iii*) and (*iv*), we get

$$a\beta=-H,\ \alpha\beta(\alpha+\beta)=G. \tag{vi}$$

Hence a, β are roots of the quadratic

$$z^2+\frac{G}{H}z-H=0.$$

α and β being thus known, the values of L and M are given by (v) and the expression $x^3+3Hx+G$ is expressed as the difference of two cubes.

We are thus left with the equation

$$\beta(x-\alpha)^3=\alpha(x-\beta)^3.$$

This gives $\sqrt[3]{\beta}(x-\alpha)=\sqrt[3]{\alpha}[(x-\beta),\ w(x-\beta),\ w^2(x-\beta)]$.

Hence the three roots are

$$-\sqrt[3]{\alpha}\ \sqrt[3]{\beta}(\sqrt[3]{\beta}+\sqrt[3]{\alpha}),\ -w\ \sqrt[3]{\alpha}\ \sqrt[3]{\beta}(w\sqrt[3]{\alpha}+\sqrt[3]{\beta}),$$
$$-w\sqrt[3]{\alpha}\sqrt[3]{\beta}(\sqrt[3]{\alpha}+w\sqrt[3]{\beta}).$$

Ex. 1. Express the general cubic expression

$$\phi_3(x)\equiv ax^3+3bx^2+3cx+d,$$

as the difference of two cubes. Hence solve the cubic $\phi_3(x)=0$.

Show that in this case α, β are the roots of the Hessain quadratic

$$\begin{vmatrix} ax+b & bx+c \\ bx+c & cx+d \end{vmatrix}=0.$$

Ex. 2. Show that if $\phi_3(x, y) \equiv ax^3+3bx^2y+3cxy^2+dy^3$, the Hessian quadratic is

$$\begin{vmatrix} \dfrac{\partial^2\phi}{\partial x^2} & \dfrac{\partial^2\phi}{\partial y\partial x} \\ \dfrac{\partial^2\phi}{\partial x\partial y} & \dfrac{\partial^2\phi}{\partial y^2} \end{vmatrix}_{y=1} =0.$$

Solve by this method the cubic $x^3-33x-18=0$.

III. Solution of the Cubic by the use of Symmetric Functions.

Let α, β, γ denote the roots of the cubic $x^3+3\text{H}x+\text{G}=0$.

The three roots are given by the expression

$^1/_3\,[\alpha+\beta+\gamma+\lambda(\alpha+w\beta+w^2\gamma)+\lambda^2(\alpha+w^2\beta+w\gamma)],\ \lambda=1,\ w,\ w^2,$

For brevity, we write

$$\text{L}=\alpha+w\beta+w^2\gamma \text{ and } \text{M}=\alpha+w^2\beta+w\gamma\ ;$$

then
$$\begin{aligned}(\lambda\text{L})^3+(\lambda^2\text{M})^3 &= \lambda^3(\alpha+w\beta+w^2\gamma)^3+\lambda^6(\alpha+w^2\beta+w\gamma)^3 \\ &= 2(\alpha^3+\beta^3+\gamma^3) \\ &\quad -3(\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta)+12\alpha\beta\gamma \\ &= 2(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma) \\ &\quad -3[(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &\qquad\qquad -3\alpha\beta\gamma]+18\alpha\beta\gamma \\ &= 27\alpha\beta\gamma=-27\text{G}, \text{ since } \alpha+\beta+\gamma=0,\end{aligned}$$

i.e.
$$\text{L}^3+\text{M}^3=-27\text{G}.$$

$$\begin{aligned}\text{Moreover } (\lambda\text{L})(\lambda^2\text{M}) &= [\lambda(\alpha+w\beta+w^2\gamma).\lambda^2(\alpha+w^2\beta+w\gamma)] \\ &= \alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha \\ &= (\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &= -9\text{H}\ ;\end{aligned}$$

i. e.
$$\text{LM}=-9\text{H}.$$

Hence L^3 and M^3 are the roots of the quadratic

$$z^2+27\text{G}z-729\text{H}^3=0.$$

Taking
$$\text{L}^3=\tfrac{27}{2}[-\text{G}+\sqrt{\text{G}^2+4\text{H}^3}],$$

and $$M^3=\tfrac{27}{2}[-G-\sqrt{G^2+4H^3}]$$

and keeping the relation $LM=-9H$ in view,
the three roots of the cubic $x^3+3Hx+G=0$ are

$$\tfrac{1}{3}\left(L-\frac{9H}{L}\right),\ \tfrac{1}{3}\left(wL-\frac{9w^2H}{L}\right) \text{ and } \tfrac{1}{3}\left(w^2L-\frac{9wH}{L}\right),$$

L being any cube root of

$$\tfrac{27}{2}[-G+\sqrt{G^2+4H^3}].$$

Ex. Solve by this method the cubic $x^3-3x+1=0$.

§ 70. Reduction of the Biquadratic to the standard form.

The general biquadratic equation is of the form

$$\phi_4(x)\equiv p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0.$$

Diminishing the roots by h, we get

$$\phi_4(x+h)\equiv\phi_0(h)x^4+4\phi_1(h)x^3+6\phi_2(h)x^2+4\phi_3(h)x+\phi_4(h)=0.$$

The second term vanishes if $p_0h+p_1=0$ *i.e.* $h=-\dfrac{p_1}{p_0}$.

The equation then, takes the form

$$p_0x^4+\frac{6}{p_0}(p_0p_2-p_1^2)x^2+\frac{4}{p_0^2}(p_0^2p_3-3p_0p_1p_2+2p_1^3)x$$
$$+\frac{1}{p_0^3}(p_0^3p_4-4p_0^2p_1p_3+6p_0p_1^2p_2-3p_1^4)=0.$$

Multiplying the roots by p_0, we get

$$x^4+6(p_0p_2-p_1^2)x^2+4(p_0^2p_3-3p_0p_1p_2+2p_1^3)x$$
$$+(p_0^3p_4-4p_0^2p_1p_3+6p_0p_1^2p_2-3p_1^4)=0.$$

Denoting $p_0p_2-p_1^2$ by H, $p_0^2p_3-3p_0p_1p_2+2p_1^3$ by G and $p_0p_4-4p_1p_3+3p_2^2$ by I,

the equation can be written in the form:

$$x^4+6Hx^2+4Gx+p_0^2I-3H^2=0.$$

This is called the standard form of the Biquadratic.

Example. Reduce to its standard form the biquadratic

$$x^4+2x^3-7x^2-8x+12=0.$$

Multiplying the roots of the given equation by 2, we get

$$x^4+4x^3-28x^2-64x+192=0. \qquad \text{(i)}$$

Increasing the roots of (i) by 1, we obtain the equation

$x^4+34x^2+225=0$ which is in the desired form.

$$\begin{array}{rrrrr}
1 & +4 & -28 & -64 & +192 \\
 & -1 & -3 & +31 & +33 \\
\hline
 & +3 & -31 & -33 & +225 \\
 & -1 & -2 & +33 & \\
\hline
 & +2 & -33 & +0 & \\
 & -1 & -1 & & \\
\hline
 & +1 & -34 & & \\
 & -1 & & & \\
\hline
 & +0 & & &
\end{array}$$

EXERCISES XXXIX

Reduce the following biquadratic equations to the standard form :—

1. $x^4+8x^3+9x^2-8x-10=0.$
2. $x^4-2x^3-12x^2+10x-3=0.$
3. $2x^4-3x^3-2x+5=0.$
4. $x^4+x^3+x^2+x+1=0.$

§ 71. Euler's Solution of the Biquadratic.

Consider the biquadratic in the standard form :

$$x^4+6Hx^2+4Gx+p_0{}^2I-3H^2=0. \qquad \text{(i)}$$

To solve this equation, assume that

$$x=p^{1/2}+q^{1/2}+r^{1/2}.$$

Squaring, we get $x^2-(p+q+r)=2(p^{\frac{1}{2}}q^{\frac{1}{2}}+q^{\frac{1}{2}}r^{\frac{1}{2}}+r^{\frac{1}{2}}p^{\frac{1}{2}}).$

Squaring again, we obtain

$$x^4-2(p+q+r)x^2-8xp^{1/2}q^{1/2}r^{1/2}+(p+q+r)^2 -4(pq+qr+rp)=0. \qquad \text{(ii)}$$

If equations (i) and (ii) be identical, we must have

$$p+q+r=-3\text{H};$$

$$p^{1/2}q^{1/2}r^{1/2}=-\frac{\text{G}}{2}, \quad \text{i.e. } pqr=\frac{\text{G}^2}{4};$$

and

$$pq+qr+rp=3\text{H}^2-\frac{p_0{}^2\text{I}}{4}.$$

Therefore, p, q, r are the roots of the cubic

$$t^3+3\text{H}t^2+\left(3\text{H}^2-\frac{p_0{}^2\text{I}}{4}\right)t-\frac{\text{G}^2}{4}=0. \qquad \text{(iii)}$$

This is known as Euler's Cubic.

We can write (iii) in the form

$$4(t+\text{H})^3-p_0{}^2\text{I}(t+\text{H})-(\text{G}^2+4\text{H}^3-p_0{}^2\text{IH})=0.$$

Writing $p_0{}^2\theta$ for $(t+\text{H})$ and $p_0{}^3\text{J}$ for $p_0{}^2\text{IH}-\text{G}^2-4\text{H}^3$, we get the cubic

$$4p_0{}^3\theta^3-p_0\text{I}\theta+\text{J}=0.$$

This is called the reducing cubic.

If θ_1, θ_2, θ_3 be the roots of the reducing cubic, then p, q, r, the roots of Euler's Cubic are given by

$$p=p_0{}^2\theta_1-\text{H},\ q=p_0{}^2\theta_2-\text{H},\ r=p_0{}^2\theta_3-\text{H}.$$

Hence $x=\pm\sqrt{p_0{}^2\theta_1-\text{H}}\pm\sqrt{p_0{}^2\theta_2-\text{H}}\pm\sqrt{p_0{}^2\theta_3-\text{H}}.$

Taking all possible combinations of signs, this appears to give eight values of x. In view of the relation

$$p^{1/2}q^{1/2}r^{1/2}=-\frac{\text{G}}{2}, \quad \text{i.e. } r^{1/2}=-\frac{\text{G}}{2p^{1/2}q^{1/2}},$$

however, it gives only four values and the biquadratic is solved.

Ex. 1. Show that for the biquadratic

$$p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0,$$

$$J = \begin{vmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \end{vmatrix}$$

Ex. 2. If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic

$$p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0,$$

show that p, q, r the roots of Euler's Cubic are given by

$$p=\frac{p_0^2}{16}(\beta+\gamma-\alpha-\delta)^2, q=\frac{p_0^2}{16}(\gamma+\alpha-\beta-\delta)^2, r=\frac{p_0^2}{16}(\alpha+\beta-y-\delta)^2;$$

and $\theta_1, \theta_2, \theta_3$ the roots of the reducing cubic are given by

$$\theta_1=\tfrac{1}{12}[(y-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)]$$
$$\theta_2=\tfrac{1}{12}[(\alpha-\beta)(\gamma-\delta)-(\beta-\gamma)(\alpha-\delta)]$$
$$\theta_3=\tfrac{1}{12}[(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)].$$

Solution. We have

$$p_0\alpha+p_1=\sqrt{p}-\sqrt{q}-\sqrt{r}, \quad p_0\beta+p_1=-\sqrt{p}+\sqrt{q}-\sqrt{r}.$$
$$p_0\gamma+p_1=-\sqrt{p}-\sqrt{q}+\sqrt{r}, \quad p_0\delta+p_1=\sqrt{p}+\sqrt{q}+\sqrt{r}.$$

Hence $p_0(\beta+\gamma-\alpha-\delta)=-4\sqrt{p}$ or $p=\dfrac{p_0^2}{16}(\beta+\gamma-\alpha-\delta)^2$.

Similarly $q=\dfrac{p_0^2}{16}(\gamma+\alpha-\beta-\delta)^2$ and $r=\dfrac{p_0^2}{16}(\alpha+\beta-\gamma-\delta)^2$.

Again $p_0(\beta-\gamma)=2(\sqrt{q}-\sqrt{r})$, $p_0(\alpha-\delta)=-2(\sqrt{q}+\sqrt{r})$.

Therefore $p_0^2(\beta-\gamma)(\alpha-\delta)=-4(q-r)=-4p_0^2(\theta_2-\theta_3)$

Similarly $p_0^2(\gamma-\alpha)(\beta-\delta)=-4p_0^2(\theta_3-\theta_1)$,

and $p_0^2(\alpha-\beta)(\gamma-\delta)=-4p_0^2(\theta_1-\theta_2)$.

Hence $(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)=4(2\theta_3-\theta_1-\theta_2)=12\theta_3$

since $\theta_1+\theta_2+\theta_3=0$.

The required relations readily follow.

Ex. 3. If in Ex. 2, $\alpha=\beta$, $\gamma=\delta$ show that two roots of Euler's cubic vanish. Hence obtain the conditions that the biquadratic may have two pairs of equal roots.

Ex. 4. If in Ex. 2, $\alpha=\beta$, show that the reducing cubic has two equal roots. Is the converse also true ?

Ex. 5. If the biquadratic $p_0x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0$ has three equal roots, show that all the roots of the reducing cubic vanish. Hence find the conditions that the biquadratic may have three equal roots.

Ex. 6. Prove that Euler's Cubic has

(*i*) all roots real and positive, (*ii*) all roots real—two negative and one positive and (*iii*) two roots imaginary and the third positive, according as the biquadratic has

(*i*) all roots real, (*ii*) all roots imaginary and (*iii*) two roots real and two imaginary.

Ex. 7. Prove that when the roots of the reducing cubic are all real the roots of the biquadratic are either all real or all imaginary. Show that the converse also is true.

Ex. 8. Prove that (*i*) when I is negative, the biquadratic has two real and two imaginary roots and (*ii*) when H and J are positive, all the roots of the biquadratic are imaginary.

§ 72. Descarte's Solution of the Biquadratic.

To solve the biquadratic

$$x^4+6Hx^2+4Gx+p_0^2I-3H^2=0,$$

assume that

$$x^4+6Hx^2+4Gx+p_0^2I-3H^2\equiv(x^2+\lambda x+\mu)(x^2-\lambda x+\nu)$$

Equating coefficients, we get

$$\left.\begin{aligned}6H&=\mu+\upsilon-\lambda^2,\\ 4G&=\lambda(\upsilon-\mu),\\ \text{and}\quad p_0^2I-3H^2&=\mu\upsilon.\end{aligned}\right\}\qquad(i)$$

Eliminating μ, υ between these equations, we get

$$(\lambda^2+6H)^2-\left(\frac{4G}{\lambda}\right)^2=4(p_0^2I-3H^2),$$

i.e.
$$\lambda^6+12H\lambda^4+4(12H^2-p_0^2I)\lambda^2-16G^2=0.\qquad(ii)$$

This is a cubic in λ^2. The last term being negative, one value of λ^2 will be real and positive. Solving (*ii*) as a cubic in λ^2, we thus obtain at least one real and positive value of λ.

The set of equations (*i*) gives then the corresponding values of μ and ν. The roots of the given biquadratic are now obtained by solving the two quadratic equations

$$x^2+\lambda x+\mu=0 \text{ and } x^2-\lambda x+\nu=0.$$

Example. Solve : $x^4-3x^2-42x-40=0.$

Assume that $x^4-3x^2-42x-40\equiv(x^2+\lambda x+\mu)(x^2-\lambda x+\nu).$

Then $\mu+\nu=\lambda^2-3,\ \mu-\nu=\dfrac{42}{\lambda}$ and $\mu\nu=-40.$

Eliminating μ, ν between these equations, we get

$$f(\lambda)\equiv\lambda^6-6\lambda^4+169\lambda^2-1764=0.$$

Now $f'(\lambda)=6\lambda^5-24\lambda^3+338\lambda,$

$$f''(\lambda)=30\lambda^4-72\lambda^2+338,\ f'''(\lambda)=120\lambda^3-144\lambda,$$

$$f^{iv}(\lambda)=360\lambda^2-144,\ f^{v}(\lambda)=720\lambda,\ f^{vi}(\lambda)=720.$$

We notice that all the derived functions are positive when $\lambda>2.$

We try the effect of the substitution $\lambda=3$ in $f(\lambda)$ as follows :—

$$\begin{array}{c|l}
 & 1+0-6+0+169+\ \ 0-1764 \\
3 & \quad +3+9+9+\ 27+588+1764 \\ \hline
 & 1+3+3+9+196+588+\ 0
\end{array}$$

Thus $f(\lambda)$ vanishes when $\lambda=3$, *i.e.* one root of $f(\lambda)=0$ is 3.

For this value of λ, we have

$$\mu+\nu=6,\ \mu-\nu=14,\ \mu\nu=-40.$$

Hence $\mu=10,\ \nu=-4.$

The roots of the given biquadratic are now obtained by solving the quadratic equations

$$x^2+3x+10=0 \text{ and } x^2-3x-4=0.$$

The first gives the roots : $\dfrac{-3\pm i\sqrt{31}}{2}$,

and the second gives the roots : -1 and 4.

EXERCISES XL

Solve the following equations by the methods of Euler and Descartes :—

1. $x^4+8x^3+9x^2-8x-10=0$. **Ans.** $1, -1, -4\pm\sqrt{6}$.
2. $x^4-2x^3-12x^2+10x+3=0$. **Ans.** $1, -3, 2\pm\sqrt{5}$.
3. $x^4+2x^3-7x^2-8x+12=0$. **Ans.** $-3, -2, 2, 1$.
4. $3x^4-23x^3+35x^2+31x-30=0$. **Ans.** $5, 3, -1, {}^2/_3$.
5. If $p_3^2=p_1^2p_4$, show that the equation
$$x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0$$
can be solved as a quadratic.

§ 73. Ferrari's solution of the Biquadratic.

Let the given biquadratic be written in the form :
$$x^4+4p_1x^3+6p_2x^2+4p_3x+p_4=0.$$

Adding $(ax+b)^2$ to the two sides, we get
$$x^4+4p_1x^3+(6p_2+a^2)x^2+(4p_3+2ab)x+(p_4+b^2)=(ax+b)^2.$$

Let us determine a and b in such a way that the left hand side of this equation may be a perfect square.

Assume that the expression on the left$\equiv(x^2+2p_1x+q)^2$.

Equating coefficients, we get
$$\left.\begin{aligned} &6p_2+a^2=4p_1^2+2q,\\ &4p_3+2ab=4p_1q;\\ \text{and}\quad &p_4+b^2=q^2\end{aligned}\right\}\qquad\text{(A)}$$

Eliminating a and b between these equations, we have
$$(2q+4p_1^2-6p_2)(q^2-p_4)=4(p_1q-p_3)^2,$$
or
$$q^3-3p_2q^2+(4p_1p_3-p_4)q-(4p_1^2p_4+2p_3^2-3p_2p_4)=0.\qquad\text{(B)}$$
This is a cubic in q and gives at least one real value for q.

Equations (A) then give the values of a and b.

The roots of the biquadratic are now obtained by solving the two quadratic equations

$$x^2+2p_1x+q=\pm(ax+b).$$

N.B.—Let α, β be the roots of the quadratic

$$x^2+(2p_1-a)x+(q-b)=0,$$

and γ, δ those of the quadratic

$$x^2+(2p_1+a)x+(q+b)=0.$$

Then since $\alpha\beta=q-b$ and $\gamma\delta=q+b$,

one value of q is $$\frac{\alpha\beta+\gamma\delta}{2}.$$

Therefore, the three roots of (B) are

$$\frac{\alpha\beta+\gamma\delta}{2}, \frac{\beta\gamma+\alpha\delta}{2}, \frac{\gamma\alpha+\beta\delta}{2}.$$

Example. Solve : $x^4+2x^3-7x^2-8x+12=0$.

Add $(ax+b)^2$ to both sides of the equation and assume that $x^4+2x^3+(a^2-7)x^2+(2ab-8)x+b^2+12\equiv(x^2+x+q)^2$.

Equating coefficients, we have

$$a^2=2q+8,\ ab=q+4,\ b^2=q^2-12.$$

Eliminating a and b from these equations, we get

$$(q+4)^2=(2q+8)(q^2-12),$$

whence $q=-4$, 4 or $-3^1/_2$.

Taking $q=4$, we have $a=4$, $b=2$.

Hence the given equation can be put in the form :

$$(x^2+x+4)^2=(4x+2)^2.$$

The roots of the biquadratic are now found by solving the two equations $x^2-3x+2=0$ and $x^2+5x+6=0$.

Therefore the roots are 1, 2. -3, -2.

§ 74. General Solution of the Biquadratic*.

The following solution gives the reducing cubic directly and would be found to be simpler than the other solutions given already.

The general biquadratic is of the form :

$$x^4+2ax^3+bx^2+2cx+d=0,$$

or

$$x^4+2ax^3=-(bx^2+2cx+d). \qquad (i)$$

Now, for all values of y, we have

$$(x^2+ax+y)^2=x^4+2ax^3+(a^2+2y)x^2+2ayx+y^2. \qquad (ii)$$

Hence, making use of (i), we get

$$(x^2+ax+y)^2=(a^2+2y-b)x^2+2(ay-c)x+y^2-d. \qquad (iii)$$

The right hand side of (iii) will be a perfect square if y satisfies the relation :

$$(ay-c)^2=(y^2-d)(2y+a^2-b). \qquad (iv)$$

This is a cubic and gives at least one real value for y.

Substituting this value of y in (iii), the two sides of (iii) become perfect squares and the equation is readily solved.

Ex. Solve : $x^4-10x^3+44x^2-104x+96=0$.

Here, we have $x^4-10x^3=-44x^2+104x-96. \qquad (i)$

For all values of y,

$$(x^2-5x+y)^2=x^4-10x^3+(2y+25)x^2-10xy+y^2. \qquad (ii)$$

Making use of (i), this becomes

$$(x^2-5x+y)^2=(2y-19)x^2-2(5y-52)x+y^2-96. \qquad (iii)$$

The right hand side of (iii) will be a perfect square if

$$(5y-52)^2=(2y-19)(y^2-96)\,;$$

i.e. if

$$y^3-22y^2+164y-440=0.$$

One root of this equation is easily seen to be 10.

Substituting 10 for y in (iii), we now have

$$(x^2-5x+10)^2=(x+2)^2.$$

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* Hansraj Gupta, Math. Student, Vol. 13, 1945, p. 31.

Solving the two quadratics

$$x^2-5x+10=\pm(x+2),$$

we get $$x=2,\ 4,\ 2\pm 2i\sqrt{2}.$$

EXERCISES XLI

Solve the following equations by each of the two methods discussed above :—

1. $x^4+8x^3+9x^2-8x-10=0.$ **Ans.** $1,\ -1,\ -4\pm\sqrt{6}.$
2. $x^4-10x^2-20x-16=0.$ **Ans.** $4,\ -2,\ -1\pm i.$
3. $x^4-2x^3-12x^2+10x+3=0.$ **Ans.** $1,\ -3,\ 2\pm\sqrt{5}.$
4. $x^4-3x^2-42x-40=0.$ **Ans.** $4,\ -1,\ \dfrac{-3\pm i\sqrt{31}}{2}$
5. Solve the equation

$$x^6-18x^4+16x^3+28x^2-32x+8=0,$$

one root being $\sqrt{6}-2.$ **Ans.** $\pm\sqrt{6}-2,\ \pm\sqrt{2},\ 2\pm\sqrt{2}.$

§ 75. Solution of Binomial Equations.

The general binomial equation is of the form :

$$x^n=a+ib,$$

where a and b are real numbers.

Let $a=r\cos\theta,\ b=r\sin\theta,$ so that $r=\sqrt{a^2+b^2}$ and θ is given by the two equations

$$\sin\theta=\frac{b}{\sqrt{a^2+b^2}},\ \cos\theta=\frac{a}{\sqrt{a^2+b^2}}.$$

Then $$x^n=r(\cos\theta+i\sin\theta)$$
$$=r[\cos(\theta+2m\pi)+i\sin(\theta+2m\pi)],$$

m being any integer.

Hence $$x=\sqrt[n]{r}\left[\cos\frac{\theta+2m\pi}{n}+i\sin\frac{\theta+2m\pi}{n}\right]$$

where $\sqrt[n]{r}$ stands for the arithmetical nth root of r ; r being a positive quantity in the above assumptions, $\sqrt[n]{r}$ always exists. Giving to m any n consecutive integral values, we obtain the n

values of x, and no more because the n values recur in periods. These are called the *n th* roots of $a+ib$.

§ 76. The *n*th Roots of Unity.

Consider the equation $x^n=1$.

Let $1=r\cos\theta$ and $0=r\sin\theta$, so that $r=1$, and θ is given by the equations $\cos\theta=1$, $\sin\theta=0$.

Hence $\theta=0$.

The given equation can be written as

$$x^n=(\cos 2m\pi+i\sin 2m\pi)$$

where m is any integer.

Therefore, $x=\cos\dfrac{2m\pi}{n}+i\sin\dfrac{2m\pi}{n}$, $m=0, 1, 2,\ldots,n-1$.

These are all the *n*th roots of unity.

Note.—Roots of the equation $x^n=1$ which do not belong to any equation of similar form and lower degree are called special roots or primitive roots of that equation, or *special nth roots of unity.*

§ 77. To consider the equation $x^n=-1$.

Let $-1=r\cos\theta$ and $0=r\sin\theta$,
so that $r=1$, and θ is given by the equations

$$\cos\theta=-1,\ \sin\theta=0.$$

Hence $\theta=\pi$, and the given equation becomes

$$x^n=\cos(2m+1)\pi+i\sin(2m+1)\pi$$

where m is any integer.

Therefore, $x=\cos\dfrac{2m+1}{n}\pi+i\sin\dfrac{2m+1}{n}\pi.$

If n is odd, one root of $x^n=-1$ is -1,

this being obtained when $m=\dfrac{n-1}{2}$.

§ 78. The Binomial equations $x^n\pm1=0$ are only particular cases of reciprocal equations discussed in an earlier chapter.

When n is even, the equation $x^n+1=0$ can be depressed to an equation of the $\frac{n}{2}$th degree and $x^n-1=0$ to an equation of the $(\frac{n}{2}-1)$th degree.

When n is odd, the equation $x^n+1=0$ can be depressed to an equation of the $\frac{n-1}{2}$th degree and $x^n-1=0$ also to an equation of the $\frac{n-1}{2}$th degree.

Example. Consider the equation $x^7+1=0$.

The roots of this equation are all included in the expression

$$\cos\frac{2m+1}{7}\pi+i\sin\frac{2m+1}{7}\pi,\ m=0, 1, 2, 3, \ldots, 6.$$

To reduce the equation to the standard form, we first divide both sides by $(x+1)$, -1 being a root of $x^7+1=0$ corresponding to the value $m=3$.

We thus have the equation

$$x^6-x^5+x^4-x^3+x^2-x+1=0. \qquad (i)$$

The roots of this equation are

$$\cos\frac{2m+1}{7}\pi+i\sin\frac{2m+1}{7}\pi,\ m=0, 1, 2, 4, 5, 6.$$

Dividing by x^3 and grouping terms equidistant from the beginning and the end, (i) becomes

$$(x^3+\frac{1}{x^3})-(x^2+\frac{1}{x^2})+(x+\frac{1}{x})-1=0. \qquad (ii)$$

Let $x+\frac{1}{x}=u$, so that

$$x^2+\frac{1}{x^2}=u^2-2,\ x^3+\frac{1}{x^3}=u^3-3u.$$

The equation (*ii*) then becomes

$$u^3-u^2-2u+1=0.$$

The roots of (*iii*) are

$$\left(\cos\frac{2m+1}{7}\pi+i\sin\frac{2m+1}{7}\pi\right)+\left(\cos\frac{2m+1}{7}\pi+i\sin\frac{2m+1}{7}\pi\right)^{-1}$$

i.e. $$2\cos\frac{2m+1}{7}\pi,\quad m=0,\ 1,\ 2.$$

EXERCISES XLII

Solve the following equations :—

1. $x^5=1.$
2. $x^8=-1.$
3. $x^4=1.$
4. $x^3=1+i.$
5. $x^2=3+4i.$

Reduce the following Binomial Equations :—

6. $x^3-1=0.$
7. $x^6+1=0.$
8. $x^5+1=0.$
9. $x^5-1=0.$

PAPERS

Delhi University B. A. Honours Examination

1948

1. (*i*) A root α of the equation $3x^3-10x^2+7x+10=0$ is connected with a root α' of the equation $x^3-x^2-17x+65=0$ by the relation $\alpha\alpha'+\alpha'-\alpha+1=0$. Using this fact, solve the two given equations completely.

(*ii*) If $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots, \alpha_n$ are the roots of the equation $x^n+nax-b=0$, show that

$$(\alpha_1-\alpha_2)\,(\alpha_1-\alpha_3)\ldots\ldots(\alpha_1-\alpha_n)=n(\alpha_1{}^{n-1}+\alpha).$$

2. Show that if p and q $(p\neq q)$ be the roots of the equation $(a_0z+a_1)\,(a_2z+a_3)=(a_1z+a_2)^2$, then the cubic equation $a_0z^3+3a_1z^2+3a_2z+a_3=0$ can be reduced to the form

$$\mathrm{A}(z-p)^3+\mathrm{B}(z-q)^3=0.$$

Hence or otherwise solve the equation

$$46z^3+72z^2+18z-11=0.$$

3. Show how to solve a biquadratic equation by resolving it into two quadratic factors.

Solve the equation $\qquad x^4+12x+3=0$

4. (*i*) Find the sum of the fifth powers of the roots of the equation $x^4-7x^2+4x-3=0$.

(*ii*) Calculate $\Sigma\,\alpha_1{}^2\,\alpha_2{}^2$ for the general equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots\ldots+p_{n-1}x+p_n=0.$$

5. State and prove Sturm's Theorem on the separation of the roots of an equation.

Find the number and situation of the real roots of the equation $x^3-3x+1=0$.

6. (*i*) Apply Newton's Method of Divisors to find the integral roots of $3x^4-23x^3+35x^2+31x-30=0$.

(*ii*) Find the positive root of the equation $x^3+x^2+x-100=0$ correct to four decimal places.

1949

1. (*i*) The sum of two roots of the equation $x^4-8x^3+19x^2+4\lambda x+2=0$ is equal to the sum of the other two. Find λ and solve the equation.

   (*ii*) Find the multiple roots of the equation
$$x^4+12x^3+32x^2-24x+4=0.$$

2. Obtain the roots of the cubic $ax^3+3bx^2+3cx+d=0$.

If the cubic $x^3+3Hx+G=0$ has roots α, β, γ, show that the cubic $x^3+9Hx^2-27(G^2+4H^3)=0$ has roots.
$$(\alpha-\beta)(\alpha-\gamma), (\beta-\gamma)(\beta-\alpha), (\gamma-\alpha)(\gamma-\beta).$$

3. (*i*) Prove that the expression
$$u \equiv z^4+6Hz^2+4Gz+a^2I-3H^2$$
can be written as the product of the two factors
$$z^2+3H+2k^2 \pm \left(2kz-\frac{G}{k}\right)$$
provided that k^2+H is a root of the equation
$$4\phi^3-a^2I\phi+a^3J=0,$$
where $G^2+4H^3=a^2(HI-aJ)$.

   (*ii*) Show that the roots of the biquadratic $ax^4+4bx^3+4dx+e=0$ have only two distinct values if
$$\frac{ad^2}{b^2e}=\frac{3bd}{bd-ae}=\pm 1,$$
and distinguish between the two cases.

4. (*i*) If α, β, γ are the roots of the equation $x^3+px+q=0$, prove that
$$\frac{\alpha^5+\beta^5+\gamma^5}{5}=\frac{\alpha^3+\beta^3+\gamma^3}{3}\times\frac{\alpha^2+\beta^2+\gamma^2}{2}.$$

   (*ii*) Apply Sturm's Theorem to the analysis of the equation $x^4-4x^3+7x^2-6x-4=0$.

5. (*i*) Find by *Horner's Method* the positive root of the equation $16x^3-20x^2-50x-375=0$.

   (*ii*) Solve $2x^3-31x^2+112x+64=0$, given that all the roots are *commensurable*.

1950

1. Calculate Sturm's remainders for the biquadratic

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0$$

and discuss the nature of the roots of that equation with the help of those remainders.

2. Solve completely the equation

$$2x^4+x^3-2x^2-4x-3=0,$$

given that two of its roots are commensurable.

Find to four decimal places the negative root of the equation

$$x^3-x^2+12x+24=0.$$

3. If $a(ax^4+4bx^3+6cx^2+4dx+e)\equiv(ax^2+2px+r)(ax^2+2qx+s)$ and $r+s=2(c-2\phi)$, find the cubic equation giving ϕ.

Solve $\quad 2x^4+6x^3-3x^2+2=0.$

4. Given that

$$x+y+z=3,$$
$$x^2+y^2+z^2=5,$$
$$x^3+y^3+z^3=7,$$

find the value of $x^4+y^4+z^4$.

If α, β, γ and δ be the roots of the equation $x^4-x+1=0$, form the equation whose roots are

$$\alpha(1+\alpha^2),\ \beta(1+\beta^2),\ \gamma(1+\gamma^2) \text{ and } \delta(1+\delta^2).$$

5. Find the condition that the roots of the equation

$$x^3+3px^2+3qx+r=0$$

may be in (*i*) A. P., (*ii*) G. P., (*iii*) H. P. Solve the equation in case (*ii*).

If $x+a_ry+a_r^2z+a_r^3t=a_r^4$ $(r=1, 2, 3, 4)$, find x, y, z and t.

1951

1. (*a*) If all the roots of the equation

$$x^n+p_1x^{n-1}+\ldots\ldots+p_n=0$$

are real and negative, show that

$$\left(\frac{p_1}{n}\right)^n \geqslant p_n.$$

(*b*) The coefficients in the equation

$$a_0 y^n + a_1 y^{n-1} + \ldots\ldots + a_n = 0$$

are connected by the relation

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots\ldots + \frac{a_n}{1} = 0,$$

show that it has at least one root between 0 and 1.

2. Prove Sturm' theorem on the separation of the real roots of an equation whose roots are unequal.

Prove that the equation,

$$x^5 - x + 16 = 0$$

has two pairs of complex roots.

3. (*a*) Find the value of the symmetric function

$$\Sigma \frac{\alpha_1{}^2 + \alpha_2{}^2}{\alpha_1 \alpha_2}$$

of the roots of the equation

$$x^n + p_1 x^{n-1} + \ldots\ldots + p_n = 0.$$

(*b*) The distances of three points A, B, C on a right line from a fixed origin 0 on the line are the roots of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0\,;$$

find the condition that one of the points A, B, C should bisect the distance between the other two.

4. Give any metood of solving the biquadratic equation

$$ax^4 + 4bx^3 + 6cx^3 + 4dx + e = 0,$$

and apply it to find the roots of

$$x^4 - 5x^2 - 6x - 5 = 0.$$

5. (*a*) If $f(x)$ be a rational integral algebraic function of x, and $\alpha, \beta, \gamma, \ldots\ldots$ be the roots of $f(x) = 0$, prove that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} + \ldots\ldots\ldots$$

Hence find the sum of the fourth powers of the roots of the equation

$$x^5+px^4+qx^2+t=0.$$

(*b*) If $f(x)$ be a rational integral algebraic function of x and the roots of $f(x)=0$ be all real, prove that the roots of

$$f(x).\ f''(x)-\{f'(x)\}^2=0$$

are all imaginary.

1952

1. (*a*) Find the relation connecting p, q, r in order that the cubic

$$x^3-px^2+qx-r=0$$

should have its roots in harmonic progression.

(*b*) If the roots $(\alpha, \beta, \gamma, \delta)$ of the equation

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

be so related that $\alpha-\delta, \beta-\delta, \gamma-\delta$ are in harmonic progression, prove that

$$ace+2bcd-ad^2-b^2e-c^3=0.$$

2. (*a*) Explain any method of solving the cubic equation

$$x^3+px+q=0.$$

(*b*) Show in any way you please that

$$x^3+x^2-2x-1=0$$

has three real roots, and determine the value of the greatest root to three decimal places.

3. State and prove a method for obtaining the sum of the nth powers of the roots of a given equation.

If $\alpha, \beta, \gamma, \ldots\ldots$ are the roots of an equation, and if S_r denotes $\Sigma\alpha^r$, prove that

$$\Sigma\alpha^3\beta^2\gamma=S_1S_2S_3-S_1S_5-S_2S_4+2S_6-S_3^2.$$

4. (*a*) If all the coefficients in the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=0$$

be whole numbers, prove that the equation cannot have a fractional root.

(*b*) If the roots of the equation $x^n-1=0$ are $1, \alpha, \beta, \gamma \ldots\ldots$ Show that

$$(1-\alpha)(1-\beta)(1-\gamma)\ldots\ldots=n.$$

5. (*a*) If the equation whose roots are the squares of the roots of the cubic

$$x^3-ax^2+bx-1=0$$

is identical with the cubic, prove that either $a=b=0$, or $a=b=3$, or a, b are the roots of $y^2+y+2=0$.

(*b*) Find the number and situation of the real roots of

$$x^6-2x^2+3x-4=0.$$

1953

1. (*a*) Increase by 7 the roots of the equation

$$3x^4+7x^3-15x^2+x-2=0,$$

and get the resulting equation.

(*b*) Solve the equation

$$x^3+3x^2+4x-10=0$$

by removing the second term.

2. Compute H and G for the cubic

$$x^3+6x^2+12x-19=0$$

and H, G, I, J for the quartic

$$2x^4+16x^3-2x^2+x-12=0,$$

where H, G, I and J have their usual meanings.

Discuss also the nature of the roots of the above quartic.

3. Define the order and weight of a symmetric fun ction, and explain their use in calculating the value of a symmetric function of the roots of an algebraic equation in terms of the coefficients.

Calculate the value of $\Sigma\alpha_1{}^2\alpha_2{}^2\alpha_3{}^2$, where $\alpha_1, \alpha_2 \ldots \alpha_5$ are the fifth roots of -1.

4. State and prove Sturm's theorem for the case of unequal roots.

Determine the integer nearest to each of the roots of

$$x^5-5x^4+9x^3-9x^2+5x-1=0.$$

5. Prove that if α be any root of

$$x^3-6x^2+11x-6=0.$$

so is $\beta=(7\alpha-13)/(3\alpha-5)$; and that a homographic relation of similar properties exists in the case of every cubic equation.

1954

1. (*a*) Find the value of the symmetric function

$$\Sigma \frac{\alpha_1{}^2+\alpha_2{}^2}{\alpha_1\alpha_2}$$

of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0.$$

(*b*) The distances of three points A, B, C on a straight line from a fixed origin O on the line are the roots of the equation

$$ax^3+3bx^2+3cx+d=0;$$

find the condition that one of the points A, B, C should bisect the distance between the other two.

2. State and prove Sturm's theorem for locating the real roots of an algebraic equation, no two of the roots being equal.

Locate the real roots of

$$x^4-8x^3+25x^2-36x+8=0.$$

3. Obtain Newton's formulae to determine the sum of the kth powers of the roots of an equation of the nth degree.

Hence express $\Sigma\alpha^2\beta^2\gamma$ in terms of the coefficients of the general equation of the nth degree, where $\alpha, \beta, \gamma, \ldots\ldots$ are its roots.

4. Discuss briefly, indicating the necessary conditions, the cases in which a biquadratic equation

$$x^4+px^3+qx^2+rx+s=0$$

is soluble by means of square roots only.

Solve in any way

$$x^4+2x^3+3x^2+4x+4.$$

5. Explain Newton's method of approximating to the numerical value of any real root of an algebraic equation.

The equation

$$x^4-3x^2+75x-10000=0$$

has a root between 9 and 10. Prove this and find its value to four significant figures.

Panjab University B.A. Papers

1948

1. (*a*) Diminish the roots of the equation

$$2x^5-x^3+10x-8=0$$

by 5

(*b*) Transform the equation

$$x^4+8x^3+x-5=0$$

into one in which the second terms is wanting.

2. Give any method of solving the biquadratic equation

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

and apply it to find the roots of

$$x^4-5x^2-6x-5=0.$$

3. Calculate to four decimal places that root of the equation

$$x^4-7x^2+18x-8=0$$

which lies between 0 and 1.

4. If $f(x)$ be a rational integral algebraic function of x, and $\alpha, \beta, \gamma, \ldots\ldots$ be the roots of $f(x)=0$, prove that

$$\frac{f'(x)}{f(x)}=\frac{1}{x-\alpha}+\frac{1}{(x-\beta)}+\frac{1}{(x-\gamma)}+\ldots\ldots\ldots$$

Hence find the sum of the fourth powers of the roots of the equation

$$x^5+px^4+qx^2+t=0.$$

1949

1. If α, β, γ be the roots of the cubic

$$x^3+px^2+qx+r=0$$

find the value of the symmetric functions

(*i*) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$ and

(*ii*) $\alpha^4+\beta^4+\gamma^4$.

2. (*a*) Give an account of Cardan's method of solution of the cubic

$$x^3+px^2+qx+r=0.$$

(*b*) Solve the equation

$$x^3+x^2-16x+20=0.$$

3. Find the values of the two roots of

$$3x^4-61x^3+127x^2+220x-520=0$$

which lie between 2 and 3 correct to five decimal places.

1950.

1. (*a*) Diminish the roots of the equation

$$2x^5-x^3+10x-8=0$$

by 5

(*b*) Transform the equation

$$x^4+8x^3+x-5=0$$

into one in which the second term is wanting.

2. (*a*) Prove that the equation $f(x)=0$, where $f(x)$ is a polynomial of degree n has n roods.

(*b*) State Descarte's' Rule of Signs regarding the nature of the roots of an equation $f(x)=0$ with real co-efficients. Apply it to discuss the nature of the roots of

$$x^4+15x^2+7x-11=0$$

3. Find the condition that the cubic equation may have

$$ax^3+bx^2+cx+d=0$$

three real roots.

Show that the equation

$x^3+x^2-2x-1=0$ has all its roots real.

4. Locate the real roots of the equation

$$x^4-7x^2+18x-8=0$$

and calculate the root which lies between 0 and 1 correct to four decimal places.

1951

1. (*a*) Establish the relations between the roots and coefficients of a general equation of the *n*th degree.

(*b*) Find the condition that the roots of the cubic

$$x^3-px^2+qx-r=0$$

may be in harmonical progression,

2. (*a*) Explain Descartes' method of solving the biquadratic equation,

$$x^4+px^2+rx+s=0$$

(*b*) Solve the equation,

$$x^4+12x-5=0.$$

3. (*a*) Prove that for an equation, all of whose coefficients are integers, the coefficient of the highest power of the variable being unity, an integral root is a divisor of the constant term.

Hence develop the proof of Newton's Method of Divisors for finding the integral roots of a given equation with integral coefficients.

(*b*) Find the commnsurable roots of

$$x^5-29x^4-31x^3+31x^2-32x+60=0.$$

4. (*a*) If α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0$$

prove that the equation in y whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha};\ \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta};\ \frac{\alpha\beta-\gamma^2}{\alpha+2\beta-\gamma}$$

is obtained by the transformation $axy+b(x+y)+c=0$.

Hence form the equation with the above roots.

1952

1. (*a*) Correct the mistakes (if any) in the following statements. For a correct statement write 'C' only.

(*i*) An equation $f(x)=0$ in which the coefficient of the first term is unity and the coefficients of the other terms rational numbers, cannot have a commensurable root which is not a whole number.

(*ii*) Incommensurable roots can be of the following three types only.

1. Roots involving $\sqrt{-1}$
2. Roots expressible as interminable decimal fractions.
3. Roots expressible as recuring decimal fractions.

(*iii*) If n be the degree of $f(x)$ and μ and μ' the number of changes of sign in $f(x)$ and $f(-x)$ respectively, then if $\mu+\mu'<n$, the equation $f(x)=0$ has exactly $n-(\mu+\mu')$ imaginary roots.

(*iv*) If two numbers a and b substituted for x in polynomial $f(x)$, give results with the same sign, no real root lies between them.

(*v*) There can be polynomials having the following factors only.

1. $(x-3)(x-5)(x-7)^2$.
2. $(x+\sqrt{-5}-3)(x-\sqrt{-5}+3)$.
3. $(x+7-5i)^2(x+7+5i)$.

(*b*) Find, in terms of the coefficients of the quartic equation

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

the value of the following symmetric function

$$(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2+(\alpha-\beta)^2(\gamma-\delta)^2,$$

where $\alpha, \beta, \gamma, \delta$ are the roots.

2. (*a*). Explain Cardan's method of solving the cubic equation

$$x^3+qx+r=0$$

(*b*) Solve the equation,

$$x^3-15x^2-33x+847=0$$

3. Find to four decimal places the root of the equation
$$x^4+4x^3-4x^2-11x+4=0$$
which lies between 1 and 2.

4. Find the condition that the roots $\alpha, \beta, \gamma, \delta$ of
$$x^4+px^3+qx^2+rx+s=0$$
should be connected by the relation $\alpha\beta=\gamma\delta$.

Hence or otherwise find the cubic whose roots are
$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta}, \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta}, \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}.$$

1953

1. (*a*) Explain Cardan's method of solving the cubic
$$x^3+qx+r=0.$$

(*b*) Prove that roots of
$$x^3-3x+1=0$$
or
$$2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}.$$

2. (*a*) The roots of the cubic
$$x^3+qx+r=0$$
are α, β, γ. Form the equation whose roots are
$$\beta^2+\beta\gamma+\gamma^2, \gamma^2+\gamma\alpha+\alpha^2, \alpha^2+\alpha\beta+\beta^2.$$

(*b*) Find the value of
$$\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta},$$
where α, β, γ are the roots of the cubic
$$x^3+px^2+qx+r=0.$$

3. (*a*) Show that all the roots of the equation
$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0$$
can be obtained when they are in arithmetical progression.

(*b*) Solve the equation
$$x^4+15x^3+70x^2+120x+64=0$$
where roots are in geometric progression.

4. Find correctly to five decimal places the cube root of 25 by Horner's process.

1954

I. (*a*) Prove that in an equation with real co-efficients, imaginary roots occur in pairs.

(*b*) Show that the equation

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\dots\dots\dots+\frac{H^2}{x-h}=k$$

has all real roots.

II. (*a*) Find the condition that the roots of

$$x^3+px^2+qx+r=0$$

may be in harmonical progression.

(*b*) Solve $6x^3-11x^2-3x+2=0$, given that its roots are in harmonical progression.

III. (*a*) Explain Descarte's method of solving the general biquadratic equation.

(*b*) Solve the equation

$$x^4-2x^2+8x-3=0.$$

VI. Calculate by Horner's method the root lying between $\frac{3}{2}$ and 2, of the equation $x^3-7x+7=0$, correct to four places of decimal.

MISCELLANEOUS QUESTIONS

1. In the equation $x^2-x-2=\varepsilon x^3$ the quantity ε is small; show that $-1+\frac{1}{3}\varepsilon-\frac{8}{27}\varepsilon^2$ is an approximation to a root, and determine the corresponding approximation to the root which is near 2. Write down an approximation to the third and large root. [*M.T. 1908*]

2. State the relations which exist between the roots and co-efficients of an equation and deduce that, if a be a real root of the cubic $x^3+px^2+qx+r=0$ of which the coefficients are real, then the other two roots are real if $p^2-4q-2pa-3a^2$ is positive or zero. [*M.T. 1909*]

3. Explain how the solution of the general biquadratic equation may be made to depend upon that of the auxiliary cubic $4\phi^3-\mathrm{I}\phi+\mathrm{J}=0$, and state the relation between the values of ϕ and the roots of the biquadratic.

Show that the roots of the biquadratic $ax^4+4bx^3+4dx+e=0$ have only two distinct values if $ad^2/b^2e=3bd/(bd-ae)=\pm 1$, and distinguish between the two cases. [*M.T. 1909*]

4. An approximate value of a root of a numerical algebraic equation being given, indicate in general terms, how to obtain a closer approximation. Show that the equation $x+x^2+x^3+x^4=5$ is satisfied, approximately, by $x=1{\cdot}0913$.
[*M.T. 1911*]

5. Prove that between every pair of consecutive real roots of the integral algebraic equation $f(x)=0$ there is an odd number of real roots of the derived equation $f'(x)=0$, and hence deduce the condition that the equation $f(x)=0$ may have equal roots.

Show that the equation $3x^3+5x^2+3x+k=0$ has two imaginary roots, whatever be the value of k. [*M.T. 1912*]

6. Show how to find the condition that a rational integral algebraic equation may have equal roots.

Find the values of a for which the equation $ax^3-9x^2+12x-5=0$ has equal roots, and solve the equation in one case.
[*M.T. 1913*]

7. If $f(x)$ be a polynomial in x, show that between consecutive real roots of the derived equation $f'(x)=0$ there occurs either one root or no root of $f(x)=0$.

Find the range of values of k for which the equation $x^4-14x^2+24x-k=0$ has all its roots real. [*M.T. 1915*]

8. Show that the squares of the roots of the equation $a_0x^n-a_1x^{n-1}+a_2x^{n-2}-a_3x^{n-3}+\ldots+(-1)^na_n=0$, are the roots of the equation $b_0x^n-b_1x^{n-1}+b_2x^{n-2}-b_3x^{n-3}+\ldots\ldots+(-1)^nb_n=0$, where $b_0=a_0^2$, $b_1=a_1^2-2a_0a_2$, $b_2=a_2^2-2a_1a_3+2a_0a_4\ldots\ldots$

$$b_r=a_r^2-2a_{r-1}a_{r+1}+2a_{r-2}a_{r+2}-2a_{r-3}a_{r+3}+\ldots\ldots\ldots$$

The equation whose roots are the squares of the roots of the cubic, $x^3-ax^2+bx-1=0$, is found to be identical with this cubic. Prove that either (*i*) $a=b=0$, (*ii*) $a=b=3$, or (*iii*) a and b are the roots of $z^2+z+2=0$. [*M.T. 1919*]

9. If m_1, m_2 be the roots of the quadratic

$$(x+p)(qx+r)-(px+q)^2=0,$$

show that the cubic equation $x^3+3px^2+3qx+r=0$ can be reduced to the form $(m_2+p)(x-m_1)^3-(m_1+p)(x-m_2)^3=0$, and thence shew how to solve the cubic.

Examine the case in which $m_1=m_2$; solving the cubic in that case, [*M.T. 1916*]

10. Prove that, if α, β are two roots of the equation $x^4+px^3+qx^2+rx+s=0$, the other two roots will be equal if $3(\alpha^2+\beta^2)+2\alpha\beta+2p(\alpha+\beta)+4q-p^2=0$. [*M.T. 1919*]

11. Prove that the multiple roots of the equation

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_n=0,$$

are all the common roots of $f(x)=0$, $f'(x)=0$.

If the a's are integers, and there is only one multiple root, show that this root must be real and rational, whatever its degree of multiplicity.

Determine by inspection of the equations

$$f(x)=0, f'(x)=0,\ldots\ldots$$

the multiple roots of $x^5-x^4-4x^2+7x-3=0$. [*M.T. 1920*]

12. Prove that, if a is an approximation to a root of an equation $f(x)=0$, then $a-f(a)/f'(a)$ is in general a closer approximation.

By applying this formula twice, or otherwise, find to three places of decimals the root near 2 of $x^4-12x+7=0$. [*M.T. 1921*]

13. Establish Newton's formulae connecting the sums of powers of the roots of an algebraic equation $x^n+p_1x^{n-1}+\ldots\ldots+p_n=0$ with the co-efficients.

Prove that if $y=x^2(\beta+\gamma)$, where α, β, γ are the roots of $x^3+qx+r=0$, then $y^3-3ry^2+(q^3+3r^2)y-r^3=0$. [*M.T. 1922*]

14. Given an algebraic equation $x^n+p_1x^{n-1}+\ldots+p_n=0$, write down equations whose roots are, (1) the roots of this equation diminished by a, (2) the roots of the original equation multiplied by b.

Find by Horner's method, to three significant figures, the root between 2 and 3 of $x^4-30x+18=0$. [*M.T. 1925*]

15. The equation $x^5-8x^4+22x^3-26x^2+21x-18=0$ has a repeated root, and another root that is rational. Solve the equation completely. [*M.T. 1925*]

16. State and prove Sturm's theorem as to the number and position of the real roots of an algebraical equation

$f\equiv a_0x^n+\ldots\ldots+a_n=0$, which may be assumed to have no repeated roots.

Shew that, if all the real roots of $f_r=0$ (f_r one of Sturm's functions) are known, it is possible by applying Sturm's process to the functions $f, f_1 \ldots f_r$ (without calculating $f_{r+1}\ldots$) to find (I) the number of real roots of $f=0$ in the interior of the interval between two consecutive roots of $f_r=0$, (2) the number of real roots greater than the greatest root of $f_r=0$, and (3) the number of real roots less than the least root of $f_r=0$, and hence to determine the number of real roots of $f=0$.

Hence or otherwise find the number of real roots of

$$x^5-10x^4+36x^3-72x^2+80x-32=0$$

and the greatest integer in each. [*M.T. 1925*]

17. If α, β are the roots of the equation $x^2-2px+q=0$ form the equation whose roots are $\alpha+\dfrac{1}{\beta}, \beta+\dfrac{1}{\alpha}$.

Given that two of the roots of $45x^4-54x^3-98x^2+150x-75=0$ are equal in absolute value but opposed in sign, complete the solution of the equation. [*M.T. 1926*]

18. Find the equation whose roots are the squares of the roots of $x^3+bx^2+cx+d=0$, and find the conditions that the squares of the roots of the given equation shall be in arithmetic progression. [*M.T. 1927*]

19. Establish a method for expressing the sum of the qth powers of the roots of

$$x^n+p_1x^{n-1}+\ldots\ldots+p_n=0 \text{ in terms of } p_1, p_2\ldots p_n$$

(q is a positive or negative integer)

Find the equation whose roots are

$$\frac{1}{\beta^2}+\frac{1}{\gamma^2},\quad \frac{1}{\gamma^2}+\frac{1}{\alpha^2},\quad \frac{1}{\alpha^2}+\frac{1}{\beta^2},$$

where α, β and γ are the roots of $3x^3-x+1=0$. [*M. T. 1927*]

20. Shew that every rational integral symmetric function of the roots of an algebraic equation can be expressed in one and only one way as a rational integral function of the coefficients.

In an equation $x^n+ax^{n-1}+bx^{n-2}+\ldots+k=0$ all the coefficients except a are fixed. How many values of a will, in general, cause the equation to have equal roots ? [*M.T. 1927*]

21. Explain and justify Horner's method of numerical approximation to the real roots of an algebraic equation.

Find to four places of decimals the real roots of

$$x^3+x+1=0.$$ [*M.T. 1928*]

22. If $f(x)$ is a polynomial of degree n, show that its derived function $f'(x)$ is $f'(x)=\dfrac{f(x)}{x-\alpha_1}+\dfrac{f(x)}{x-\alpha_2}+\ldots\ldots\ldots+\dfrac{f(x)}{x-\alpha_n}$.

where $\alpha_1,\ldots,\alpha_n$ are the roots of $f(x)=0$, which may or may not be distinct but may be assumed to be all real. Use this identity to prove

(1) That between every pair of distinct consecutive roots of $f(x)=0$ there lies one and only one root of $f'(x)=0$.

(2) Any root which occurs m times in $f(x)=0$ occurs $(m-r)$ times in $f^{(r)}(x)=0$.

Find the multiple roots of $x^4-2x^3-11x^2+12x+36=0$. [*M.T. 1928*]

23. If all the roots of the equation $x^n+p_1x^{n-1}+\ldots+p_n=0$ are rational and negative, show that

$$\text{(i) } \left(\frac{p_1}{n}\right)^n>p_n, \quad \text{(ii) } \left(\frac{p_1}{n}\right)^r>p_r\frac{\lfloor r \lfloor n-r}{\lfloor n}$$

[*M.T. 1929*]

24. Establish a method of solving algebraically the general equation of the fourth degree.

Discuss the reality of the roots of

$$16x^4+24x^2+16kx+9=0$$

for all real values of k, and solve the equation when $k=\sqrt{2}$. [*M.T. 1929*]

25. State without proof Sturm's method of determining the positions of the real roots of an equation $f(x)=0$.

It may be assumed that all the roots are unequal.

Prove that $x^4-18x^3+4dx+9=0$ has four real roots, if $d^4\leqslant 1728$. [*M.T. 1930*]

26. If $f(x)$ is a polynomial with real coefficients for which the real roots of the equation $f'(x)=0$ are all known, shew how to determine the number of real roots of the equation $f(x)=0$.

Discuss the reality of the roots of the equation

$$x^4+4x^3-2x^2-12x+a=0$$

for all real values of a. [*M.T. 1930*]

27. Shew that if p and q ($p\neq q$) be the roots of the equation $(a_0z+a_1)(a_2z+a_3)=(a_1z+a_2)^2$, then the cubic equation $a_0z^3+3a_1z^2+3a_2z-a_3=0$ can be reduced to the form

$$A(z-p)^3+B(z-q)^3=0.$$

Solve the equation $46z^3+72z^2+18z-11=0$. [*M.T. 1930*]

28. The roots of the equation $16x^4-64x^3+56x^2+16x-15=0$ are known to be in arithmetical progression; solve the equation. [*M.T. 1931*]

29. Find the equation whose roots are given by the formula $y_1=x_1-x_1^2+(x_1^2+x_2^2+_3^2)$, $(i=1, 2, 3)$ where x_1, x_2, x_3 are the roots of the equation $x^3-x^2+4=0$.

Solve the equation $6x^4-3x^3+8x^2-x+2=0$, being given that it has a pair of roots whose sum is zero. [*M.T. 1932*]

30. If $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ are the roots of the equation

$$x^n + nax - b = 0, \text{ shew that}$$

$$(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\ldots(\alpha_1 - \alpha_n) = n(\alpha_1{}^{n-1} + a).$$

Form the equation whose roots are $\alpha_1{}^{n-1}, \alpha_2{}^{n-1} \ldots \alpha_n{}^{n-1}$ and shew that the product of the squared differences of the roots of the first equation is

$$(-1)^{1/2(n^2+n-2)} n^n[(n-1)^{n-1}a^n + b^{n-1}] \qquad [M.T. \ 1934]$$

31. If the equation $x^4 - 4ax^3 + 6x^2 + 1 = 0$ has a repeated root ζ, shew that $3a = \dfrac{\zeta^2 + 3}{\zeta}$.

Hence or otherwise prove that there is only one positive a giving a repeated root, and that this value of a is

$$\left(\frac{4}{3}\right)^{\frac{3}{4}}. \qquad [M.T. \ 1936]$$

32. Find the range of values of k for which the equation

$$x^3 - 6x^2 + 9x + k = 0$$

has three real roots.

Taking $k = -1$ evaluate the largest root of the equation correct to two places of decimals. (*M.T. 1940*).

33. If a, b, c, d are the roots of the quartic equation

$$x^4 + px^3 + qx + r = 0$$

and $S_n = a^n + b^n + c^n + d^n$, prove that

(*i*) $S_n + pS_{n-1} + qS_{n-3} + rS_{n-4} = 0$,

(*ii*) $S_4 = p^4 + 4pq - 4r$ (*M.T. 1941*)

34. Prove that the cubic expression $ax^3 + 3cxy^2 + dy^3$ can in general be expressed uniquely in the form

$$(lx + my)^3 + (l^1x + m^1y)^3$$

Hence, or otherwise, find the real root of the equation $x^3 - 18x - 30 = 0$ correct to two places of decimals. (*M.T. 1942*)

35. Prove that, if $\lambda^3 = 1$, one root of the equation

$$\begin{vmatrix} a-x & b & c \\ c & a-x & b \\ b & c & a-x \end{vmatrix} = 0$$

is $x=a+\lambda b+\lambda^2c$, and hence solve the equation.

Obtain in a similar form the equation whose roots are the squares of the roots of this equation. *(M.T. 1943)*

36. Prove that the remainder when a polynomial $f(x)$ is divided by $(x-\alpha)$ is $f(\alpha)$.

Prove that, when $f(x)$ is divided by $(x-\alpha)(x-\beta)$, where $\alpha\neq\beta$, the remainder is

$$\frac{(x-\beta)\,f(\alpha)-(x-\alpha)\,f(\beta)}{\alpha-\beta}.$$

Obtain an expression for the remainder when $f(x)$ is divided by $(x-\alpha)^2$. *(M.T. 1943)*.

37. (*i*) If the equations

$$ax^3+bx^2+c=0 \text{ and } bx^3+cx^2+a=0$$

have a common root, prove that

$$(a^2-bc)^3=(b^2-ac)(c^2-ab)^2.$$

(*ii*) If α, β, γ are the roots of the equation

$$x^3+px+q=0,$$

prove that the roots of the equation

$$x^3+4px^2+5p^2x+2p^3+q^2=0$$

are
$$\beta^2+\gamma^2,\ \gamma^2+\alpha^2,\ \alpha^2+\beta^2.$$ *(M.T. 1944)*

38. If a, b, c, d are the roots of the equation

$$px^4+qx^3+rx^2+x+1=0,$$

prove that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}=0$$

(M.T. 1945)

39. If α, α' are the roots of the equation $ax^2+2bx+c=0$ and β, β' are the roots of the equation $py^2+2qy+r=0$, prove that $\alpha\beta, \alpha\beta', \alpha'\beta, \alpha'\beta'$ are the roots of the equation

$$(apz^2-2bqz+cr)^2-4(ca-b^2)(rp-q^2)z^2=0.$$

(M. T. 1945)

40. Find the range of values of the co-efficient p for which the equation

$$2x^3+9x^2+12x+p=0$$

has three real unequal roots.

Determine, correct to one decimal place, the real root of equation

$$x^3+12x^2+9x+2=0. \qquad (M.T.\ 1946)$$

41. It is given that the sum of two roots of the equation

$$x^4-8x^3+19x^2+px+2=0$$

is equal to the sum of the other two. Find the value of p, and all the roots of the equation. (*M.T. 1948*)

42. Prove that, if ζ is an approximation to a root of $f(x)=0$, then, in general,

$$\zeta_1=\zeta-\frac{f(\zeta)}{f'(\zeta)}$$

is a better approximation.

If $f(\zeta)$ is negative and $f'(x)$, $f''(x)$ have constant signs between ζ and ζ_1 what must be the sign of $f''(x)$ in that interval for the roots to lie between ζ and ζ_1 ?

Prove that the equation

$$2x^5-2x=61$$

has a root between 2 and 2·00633. (*M.T. 1949*)

43. Find a condition on the co-efficients in order that the form

$$f\equiv a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4$$

may be expressed as

$$\lambda(x+\theta)^4+\mu(x+\phi)^4.$$

If the condition is satisfied, find a quadratic for θ, ϕ, and prove that the roots of $f=0$ form a harmonic range. (*M.T. 1950*)

44. If z is a complex number and $\bar{z}$ is its complex conjugate, prove that $|z|=1$ if and only if $z^{-1}=\bar{z}$.

If a, b, c are complex numbers such that $a\neq0$ and $a\bar{a}$

$\neq c\bar{c}$, prove that a necessary and sufficient condition that one of the roots of the equation

$$az^2+bz+c=0$$

should have modulus 1 is that

$$|\bar{a}b-\bar{b}c|=|a\bar{a}-c\bar{c}|. \qquad \textit{(M.T. 1953)}$$

45. If α, β, γ are the roots of the equation

$$x^3-s_1x^2+s_2x-s_3=0,$$

find a necessary and sufficient condition in terms of s_1, s_2, s_3 that two of the roots should be equal.

Prove that, if $\beta=\gamma\neq\alpha$,

$$\beta=(s_1s_2-9s_3)/(2s_1^2-6s_2). \qquad \textit{(M.T. 1953)}$$